ECON326 - THE ECONOMICS OF FINANCIAL MARKETS

LECTURE NOTES

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INTRODUCTION

The following pages collect the material for the course in Financial Markets that I have taught at several institutions, including University of Southampton, University College London and Nazarbayev University. This is not to be intended as a first draft of a book or notes to be published,[1] but as material that has been prepared in order to help students to follow the course. There might be (and there will be) typographic or other mistakes, but they are not accepted as a basis for any kind of appeal against marks allocated in the exam. Students are welcome to spot errors and typos.

This reader is not a substitute for lectures. The material in class might change or expanded and hence students are expected to attend lectures. It is students' responsibility to compare the contents of this reader with the contents of lectures.

Problem sets will be distributed during the semester. Exercising in solving the problems is the only way to check whether the material has been understood.

Students are welcome to ask questions during the lectures and office hours. In the case they are unable to come during office hours, they are entitled to ask for an appointment by e-mail (`giulio.seccia@nu.edu.kz`).

[1] In fact, part of the material in Lecture 6 to 10 is hardly original being based on different textbooks, e.g. Copeland and Weston *Financial Theory and Corporate Policy*, Addison-Welsey, 1992 and Elton E. and M. Gruber, *Modern portfolio theory and investment analysis*, Wesley, 1995.

Lecture 1

CONSUMPTION CHOICE OVERTIME

1.1 Introduction

In this lecture we study the problem of consumption allocation across time in a very simple economy composed by one agent only. We will see that borrowing and lending opportunities allow the transfer of consumption across time. This is the first role performed by financial markets. Later we will see how in the case of uncertainty the trade of assets allows the transfer of consumption across nature contingencies.
We proceed as follows: we first analyze the consumption allocation when neither financial markets nor production opportunities exist. We then introduce production opportunities only, then financial markets only. We analyze and compare consumption allocation in each instance. Finally, we allow the agent to choose to allocate consumption across time by production and by lending/borrowing opportunities. The last section presents the Fisher Separation Theorem that identifies the fundamental allocation rule.

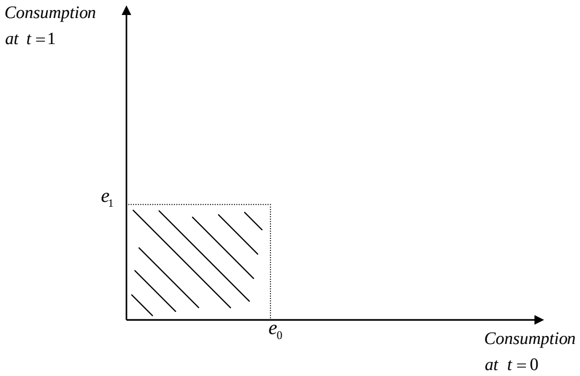
1.2 Description of the economy

We start by considering a very simple economy. Economic activity extends over two periods, $t = 0, 1$. One good is consumed in each period. There is one agent in the economy.
Denote by e_0 and e_1 the *endowment* of the agent in the first and second period, respectively.
Denote by x_0 and x_1 the *consumption* of the agent in the first and second period, respectively.

How will the agent allocate consumption across time?: We need two pieces of information in order to proceed:
1) The opportunity set of the agent;
2) The preferences of the agent.

1.2.1 Opportunity sets

Case a): *No storage, no production and no financial markets.*



We can write the budget set as follows: $\{(x_0, x_1) : x_0 \leq e_0,\, x_1 \leq e_1\}$.

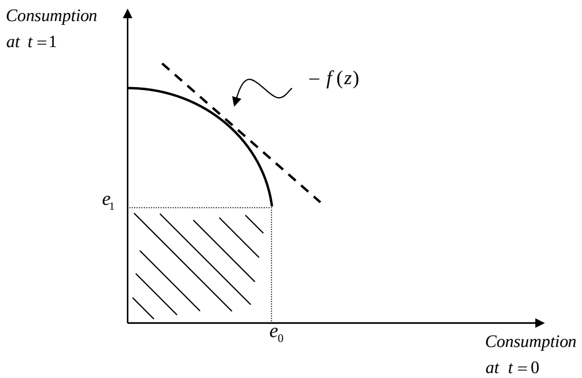
Case b): *Production without financial markets.*
Suppose now that the agent has access to a production technology. He can use part of his endowment today and produce some of the good tomorrow. Let z be the amount of good he uses today as input. Let the production function be described by the function $f : z \to f(z)$.

Assumption 1.1. $f(0) = 0$, $\frac{\partial f(z)}{\partial z} \geq 0$; $\frac{\partial^2 f(z)}{\partial z^2} \leq 0$.

Exercise 1.2.1. *1. Explain in words the meaning of Assumption 1.1.*

2. Check that the function $f(z) = \sqrt{z}$ does satisfy Assumption 1.1 but the function $f(z) = \ln z$ does not.

Graphically:



The budget set of the individual is now given by:

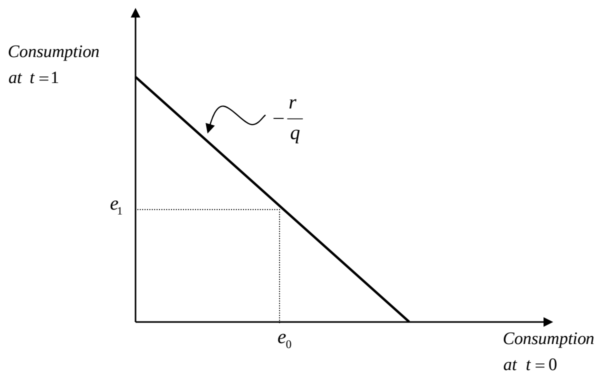
$$\begin{aligned}\{(x_0, x_1) :& x_0 + z \leq e_0, \\ & x_1 \leq e_1 + f(z)\}.\end{aligned}$$

Case c) *Financial markets without production.*
Suppose now that the agent can exchange one unit of the good versus a contract promising the holder the delivery of a fixed amount r (the *return* of the contract) of the good tomorrow versus a payment (the *price* of the contract), say q, today. This contract is equivalent to a *bond* or debt. The buyer or holder of a contract is a *lender* (he exchanges consumption today versus a promise of delivery of consumption tomorrow). The seller of the contract is a *borrower.* Notice that the seller is holding a negative amount of bonds. Throughout the course we will assume that all promises are kept, *i.e.,* there is no default on contracts.

Assumption 1.2. *The returns from borrowing and lending are the same.*

Let y be the amount of contracts the agent decides to buy (or to sell, if y is negative) in the financial market.
The budget constraint of the individual with access to financial markets is given by:

$$\begin{aligned}\{(x_0, x_1) : x_0 + qy &\le e_0,\\ x_1 &\le e_1 + ry\}\end{aligned}$$



Exercise 1.2.2. *How does the budget set change if, given q, r increases (decreases)?*

Exercise 1.2.3. *Explain the differences between a production technology and a financial markets in terms of opportunities of transferring goods across time.*

Exercise 1.2.4. *Show that storage is a simple type of production technology.*

Case d) *Production and financial markets*

With both production and financial markets the budget constrain becomes:

$$\begin{aligned}\{(x_0, x_1) : x_0 + z + qy &\le e_0,\\ x_1 &= e_1 + f(z) + ry\}\end{aligned} \tag{1.1}$$

The following assumption guarantees that for low levels of investments, the production technology delivers a higher return than the financial markets. It guarantees an interior solution for the production choice to the optimization problems that we are about to study.

Assumption 1.3. $f'(0) > \frac{r}{q}$.

1.2.2 Wealth Maximization

Let us now assume that the agent has a certain amount of good today, say e_0, that he wants to invest either in the production process or in the financial market. Suppose he cares only about his final wealth. If this is so, then he will choose between input (z) and investment (y) so that to solve the following problem:

$$\begin{aligned}&\max_{z,y} \quad f(z) + ry\\ &\text{s.t.} \qquad z + qy = e_0\end{aligned}$$

with solution: $f'(z^*) = \frac{r}{q}$ or:

$$\boxed{z^* = f'^{-1}\left(\frac{r}{q}\right)}$$

The equality says that at the optimum, the marginal productivity of capital must equal the *rate* of return $\frac{r}{q}$.

Example 1. *: Let $f(z) = \sqrt{z}$, $r = 1.05$ and $q = 1$. Then the optimal choice of production investment is given by:*

$$\frac{1}{2\sqrt{z}} = 1.05$$

or: $z = (2.1)^{-2}$.

1.2.3 Preferences

Assumption 1.4. *Preferences are described by a continuous, strictly monotonic, strictly concave and time separable utility function, say:*

$$U = u(x_0) + u(x_1).$$

This assumption gives well-behaved strictly convex indifference curves (*why*?).

Exercise 1.2.5. *Explain in words what the Assumption 1.4 means.*

1.3 Optimal consumption/investment choice

1.3.1 An example

Suppose that the preferences of the consumer are represented by the utility function:

$$\ln x_0 + \ln x_1.$$

Let (e_0, e_1) the endowment of the individual at time $t = 0, 1$ and suppose the agent owns the production technology represented by the production function $f(z)$.

Exercise 1.3.1. *Show that without financial markets, the optimal production choice (z) depends upon consumer's preferences.*

Assume that the agent has the opportunity to lend and borrow with return r and let y be the amount of consumption good today that he borrows/lends.

The agent has to decide how to allocate consumption between today and tomorrow. In order to do that, he has to decide how much to invest in the production process and how much to invest in the financial market.

The agent maximization problem can be represented as follows:

$$
\begin{array}{ll}
\max & \ln x_0 + \ln x_1 \\
\text{s.t.} & x_0 \leq e_0 - qy - z; \\
& x_1 \leq e_1 + ry + f(z) \\
& z \geq 0.
\end{array}
$$

Assumption 1.3 guarantees that the last constraint is satisfied. Substitute $x_0 = e_0 - y - z$ and $x_1 = e_1 + ry + f(z)$ in the objective function and take the first order conditions of the problem:

$$
\begin{array}{ll}
z: & -\dfrac{1}{e_0 - qy - z} + f'(z)\dfrac{1}{e_1 + ry + f(z)} = 0; \\
y: & -\dfrac{q}{e_0 - qy - z} + \dfrac{r}{e_1 + ry + f(z)} = 0.
\end{array}
$$

Rearranging the terms obtain:

$$
f'(z^*) = \frac{r}{q},
$$

where the star denotes the optimal choice. This is the same we obtained in (1.3), where we miximized wealth and not preferences. This is an example of the Fisher separation theorem, one of the most important *irrelevance propositions* in finance: preferences are irrelevant in the investment decision. Remember that this is true only under the above hypothesis of "perfect" financial markets.

Exercise 1.3.2. *What is the optimal input choice $z*$ if the production function is given by $f(z) = \sqrt{z}$?*

1.3.2 Solving the agent's problem in the general case

The agent's optimal problem is given by:

$$
\begin{array}{ll}
\max & U(x_0, x_1) \\
\text{s.t.} & x_0 \leq e_0 - qy - z; \\
& x_1 \leq e_1 + ry + f(z); \\
& z \geq 0.
\end{array}
\tag{1.2}
$$

Writing the Lagrangean:

$$
\mathcal{L} = U(x_0, x_1) + \mu_0(e_0 - qy - z - x_0) + \mu_1(e_1 + ry + f(z) - x_1) + \mu_2 z,
$$

where μ_0, μ_1, μ_2 are the Lagrangean multipliers for the three constraints in the original problem. The *F.O.C.* of the problem are the following:

$$
\begin{aligned}
x_0 : &\quad \frac{\partial U}{\partial x_0} = \mu_0; \\
x_1 : &\quad \frac{\partial U}{\partial x_1} = \mu_1; \\
z : &\quad -\mu_0 + \mu_1 \frac{\partial f}{\partial z} + \mu_2 = 0; \\
y : &\quad -\mu_0 q + \mu_1 r = 0; \\
\mu_0 : &\quad x_0 \leq e_0 - qy - z; \\
\mu_1 : &\quad x_1 \leq e_1 + ry + f(z); \\
\mu_2 : &\quad z \geq 0.
\end{aligned}
$$

By assumption 1.3 the last constraint holds with inequality: $z > 0$ (what does this say about μ_2?). From the first two F.O.C. obtain:

$$\frac{\partial U}{\partial x_0}\left(\frac{\partial U}{\partial x_1}\right)^{-1} = \frac{\mu_0}{\mu_1}.$$

From the forth F.O.C., obtain:

$$\frac{\mu_0}{\mu_1} = \frac{r}{q}.$$

and hence:

$$\frac{\partial U}{\partial x_0}\left(\frac{\partial U}{\partial x_1}\right)^{-1} = \frac{r}{q},$$

i.e.: the Marginal Rate of Substitution ($-\frac{\partial U}{\partial x_0}\left(\frac{\partial U}{\partial x_1}\right)^{-1}$) equals $-\frac{r}{q}$.
Also obtain the fundamental production decision rule :

$$\frac{\partial f}{\partial z} = \frac{r}{q}.$$

1.3.3 Fisher Separation Theorem

Proposition 1.3.1. ***Fisher Separation Theorem:*** *If borrowing and lending rates are the same, the production decision is governed by the maximization of wealth, independently of individual preferences.*

Note: The theorem does not say that preferences are irrelevant for the consumption decision, neither for the borrowing or lending choice. Preferences do still determine these quantities, but not the optimal production choice.

Exercise 1.3.3. *Show the Fisher Separation Theorem graphically.*

Exercise 1.3.4. *Explain the following statement: Fisher Separation Theorem has two corollaries that are central in corporate finance: shareholders' unanimity and separation of ownership and control.*

1.3.4 Value of the firm

The value of the firm can be easily computed by looking at the budget constrain in (1.2). Substituting away y and reducing it to one constrain obtain:

$$x_0 + z + \frac{q}{r}x_1 - \frac{q}{r}e_1 - \frac{q}{r}f(z) = e_0$$

or:

$$\overbrace{x_0 + \frac{q}{r}x_1}^{\text{consumption present value}} \quad = \quad \overbrace{e_0 + \frac{q}{r}e_1}^{\text{wealth present value}} \quad + \quad \overbrace{\frac{q}{r}f(z) - z}^{\text{value of the firm}}$$

Substituting the optimal value z^* obtain the value of the firm at the optimum:

$$\frac{q}{r}f\left(f'^{-1}\left(\frac{r}{q}\right)\right) - f'^{-1}\left(\frac{r}{q}\right) \tag{1.3}$$

1.3.5 Equilibrium in a simple dynamic economy

Exercise 1.3.5. *Consider a simple two periods, $t = 0, 1$, one good, two agents, $h = 1, 2$, economy. Each agent $h = 1, 2$ is endowed with good e_t^h at each period t. Assume that agents' preferences are represented by the utility function $\ln x_0 + \ln x_1$ and that in the first period agents can exchange a bond at price q and given return r in the second period. What is the equilibrium in this economy?*

Appendix: Marginal Rate of Substitution and optimal consumption choice

Consider the standard static, two-good choice of a consumer with wealth M:

$$\max_{x_1,x_2} U(x_1, x_2)$$
$$\text{s.t. } p_1x_1 + p_2x_2 = M.$$

Solve the budget constrain for x_2 and substitute into the utility function. The problem becomes:

$$\max_{x_1} \; U(x_1, \frac{M}{p_2} - \frac{p_1}{p_2}x_1),$$

with first order condition (with respect to the only control variable x_1) given by:

$$\frac{\partial U}{\partial x_1} + \frac{\partial U}{\partial x_2}\frac{\partial x_2}{\partial x_1} = 0.$$

Since from the budget constrain $\frac{\partial x_2}{\partial x_1} = -\frac{p_1}{p_2}$

$$\text{MRS}(x_1, x_2) = -\frac{\frac{\partial U}{\partial x_1}}{\frac{\partial U}{\partial x_2}} = -\frac{p_1}{p_2}.$$

Lecture 2

CHOICE UNDER UNCERTAINTY

2.1 Introduction

In Lecture 1 we have analyzed the problem of an agent that has to allocate consumption across two periods. In particular, we assumed that the agent knew exactly his future wealth and future production output, *i.e.,* he was facing no economics uncertainty. The purpose of the following lecture is to introduce economic choice in an uncertain environment. Uncertainty can refer to fluctuations of the fundamental parameters describing the economy: endowments, preferences and the production technology[1].
In this lecture we will give a definition and a measure of the attitude of agents toward risk and how this can affect agents' choices. This is essentially related to agents' preferences.
Remark (very important): in order to simplify the presentation, we will assume that the utility of an agent is defined over his wealth, *i.e.,* we will deal with the agent's indirect utility. The definitions that we will give (attitude toward risk, concavity etc.) have logical analogues when the utility is defined over consumption bundles.

2.2 Gambling and expected utility

Example: Consider an individual facing the following simple lottery. A ticket for the lottery costs £40. There are only two outcomes, head or tail, each with equal probability $\frac{1}{2}$. If head results, the lottery pays[2] $£40 - £24 = £16$, otherwise it pays $£40 + £24 = £64$. This is equivalent to say that the individual faces a lottery (or a bet), call it $\tilde{Z}$, at no cost, with two, equally likely outcomes: $Z_1 = -£24$ and $Z_2 = £24$. We can denote this lottery as $\tilde{Z} = (\frac{1}{2}, -24; \frac{1}{2}, 24)$.
The individual has to decide whether to keep his wealth ($W = £40$) or to enter the lottery and take the risk. This is equivalent to the situation where he has to decide whether to enter the zero price lottery $\tilde{Z}$ described above. The first thing we want to understand is how to classify the agent's attitude toward risk by looking at his preferences. In order to proceed we need some definitions.

[1] Economists call this type of uncertainty "intrinsic" in order to distinguish it from the uncertainty related to other parameters that, nevertheless, may affect economic activity: for instance, price levels, monetary policies and, most interestingly, agents' beliefs. This type of uncertainty is called "extrinsic". How "extrinsic" uncertainty affects equilibrium allocations has been a very important area of interest in the recent economic theory. Don't worry: we will only study cases with intrinsic uncertainty.

[2] If you prefer you can think that the lottery pays in consumption good so that you can express utility over the good itself.

Definition 2.2.1. *Let $\tilde{Z}$ be a random variable taking value $Z_s : s = 1, \ldots, S$ each with probability π_s. The* ***expected*** *value of the variable is given by:*

$$E(\tilde{Z}) = \sum_s \pi_s Z_s.$$

The ***actuarial value of a gamble*** *is its expected payoff.*

Example: In the above case the **actuarial value of the gamble** is given by:

$$\frac{1}{2} \times (£16) + \frac{1}{2} \times £64 = 40.$$

that is equal to the price of the game. The equivalent representation would give:

$$\frac{1}{2} \times (-£24) + \frac{1}{2} \times £24 = 0.$$

Suppose now that the utility of the individual with initial wealth W is given by $u(W + \tilde{Z})$ where $W + \tilde{Z}$ is the consumer's total income, given by his initial wealth and the outcome of the lottery. We call the **expected utility** the weighted average of the utility from each prize of the gamble, where again the weights are given by the probability to obtain each prize. In this case:

$$\frac{1}{2} \times u(16) + \frac{1}{2} \times u(64).$$

The following is a more general definition of *expected utility*:

Definition 2.2.2. *Let $\tilde{x}$ be a random variable taking value $x_s : s = 1, \ldots, S$ each with probability π_s. Let $u(x_s)$ the utility of an agent at each realization of the random variable $\tilde{x}$. The* ***expected utility*** *function (in this case also called* ***von Neumann-Morgerstern utility****) is given by the following value:*

$$Eu(\tilde{x}) = \sum_{s=1}^{S} \pi_s u(x_s).$$

In the example above $x_s = W + Z_s$. If the agent's utility function is given by the square root of consumption, then in our case the agent's expected utility from entering the gamble is given by:

$$\frac{1}{2} \times \sqrt{16} + \frac{1}{2} \times \sqrt{64} = 2 + 4 = 6$$

2.3 Risk aversion

Consider[3] the lottery presented in the previous section. Remark that the expected value of the lottery is zero: we call this type of lotteries **actuarially neutral** and we say that the agent faces a **fair gamble**.

We say that an agent is **risk averse** if his expected utility is such that he does not accept fair gambles (so his *utility* from *not* accepting this type of gambles is higher than his *expected utility* from entering them). If he is willing to accept them, then we say he is **risk lover** and if he is indifferent, we say he is **risk neutral**. Alternatively: an agent is risk averse if the utility from an amount equivalent to the expected value of the **fair gamble** is higher than the expected

[3]See also Varian, *Intermediate Microeconomics: A modern approach*, Norton

utility of the gamble.

Example: Consider the lottery in the previous section. The agent is said to be risk averse if

$$u(\frac{1}{2} \times 16 + \frac{1}{2} \times 64) > \frac{1}{2} \times u(16) + \frac{1}{2} \times u(64).$$

If the equality holds he is said to be risk neutral and if the inequality is reversed then he is said to be risk lover.

Definition 2.3.1. *Let $\tilde{Z}$ a random payoff of an actuarially neutral lottery, promising a prize Z_s, $s = 1, \ldots, S$ with probability π_s.*
The attitude toward risk of an agent with initial wealth W and utility $u(W + \tilde{Z})$ can be defined as follows:

*if $u(E(W + \tilde{Z})) > E[u(W + \tilde{Z})]$ then the agent is said to be **risk averse**;*

*if $u(E(W + \tilde{Z})) = E[u(W + \tilde{Z})]$ then the agent is said to be **risk neutral**;*

*if $u(E(W + \tilde{Z})) < E[u(W + \tilde{Z})]$ then the agent is said to be **risk lover**.*

2.4 Risk aversion and strictly concave utility functions

Definition 2.4.1. ***Concave utility functions*** *Consider two distinct income levels, say M_1 and M_2. Then, a utility function u is said to be concave if and only if, for any number λ between 0 and 1, the following inequality holds:*

$$u(\lambda M_1 + (1 - \lambda)M_2) \geq \lambda u(M_1) + (1 - \lambda)u(M_2).$$

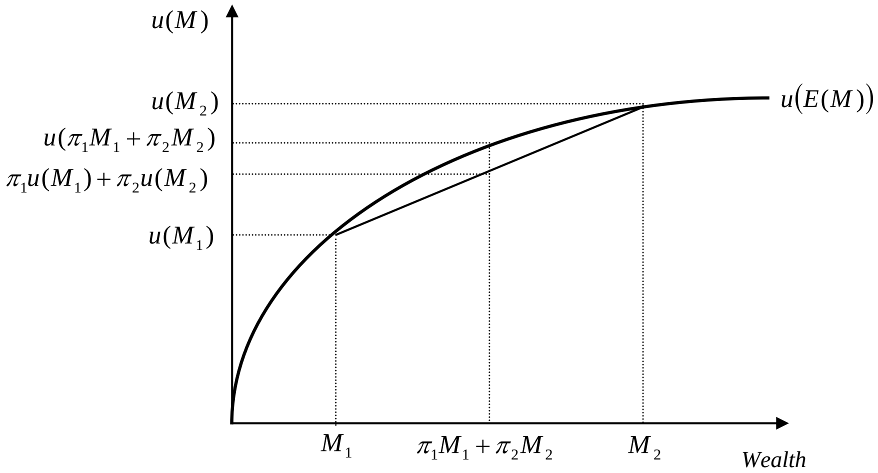
If the strict inequality holds, then the utility is said to be strictly concave.

Exercise 2.4.1. *: Show, by an numerical example, that the following utility functions are strictly concave:*

$u(M) = \sqrt{M}$

$u(M) = \ln M.$

You can easily recognize that when an agent has a concave utility function, then he is risk averse. In fact, he will always prefer the actuarial value of a fair gamble than to enter the gamble. A graph best represents the idea.



Remark: Concavity of the utility function implies that the marginal utility of wealth is decreasing. So, the potential loss of £24 is not compensated by the potential equally probable gain of £24.

2.5 Risk premium

Definition 2.5.1. ***Certainty equivalent wealth****: level of wealth that the individual would accept with certainty if an actuarially neutral gamble were removed.*

Example: Suppose that the agent playing the lottery in section 2.2 has utility given by $\sqrt{M}$. Computing his expected utility from the lottery you obtain $\frac{1}{2}\sqrt{16}+\frac{1}{2}\sqrt{64}=6$. So, the certainty equivalent wealth is given by £36. This is less than the initial wealth of the agent (£40).

Definition 2.5.2. ***Risk premium****:*

Risk premium= expected wealth - certainty equivalent wealth.

Exercise 2.5.1. *Compute the risk premium according to definition 2.5.2 in the example above.*

Remark: In this course we will always assume agents' utility functions to be strictly concave and hence we will always deal with risk averse agents.
A more rigorous definition of risk premium, due to Markowitz, is as follows:

Definition 2.5.3. *The* ***(Markowitz) risk premium*** *is given by the value $P(W, \tilde{Z})$ such that:*

$$E[u(W+\tilde{Z})] = u[W + E(\tilde{Z}) - P(W, \tilde{Z})].$$

Notice that the risk premium is measured in wealth. This is very useful. Comparing utility levels across agents facing the same risk is meaningless (*why?*) but it is perfectly fine to compare their risk premia.

2.6 Arrow-Pratt measures of risk aversion.

The shape of the utility function tell us in which category an individual falls in term of his/her risk attitude: averse, neutral or risk lover. But, given the individual's risk attitude, the aversion to accepting a certain risk will also depend on the individual's wealth before taking the gamble. Intuitively, given risk and risk attitude we would expect a wealthy individual to "dislike" taking a certain risk less than a less wealthy one. "Disliking risk" here can be quantified in terms of the compensation an individual demands in order to take an actuarially neutral gamble. An obvious question that arises is how to measure individuals' risk aversion. De Finetti (1952), Arrow (1963) and Pratt (1964) independently developed the analysis of risk aversion. We now derive two very useful measures of risk aversion as developed by Arrow and Pratt.

Suppose that a certain individual has wealth W. Consider now a small, actuarially neutral lottery $\tilde{Z} = (Z_s, \pi_s : s = 1, 2, \ldots, S)$ and the associated risk premium $P(W, \tilde{Z})$ faced by an agent with initial wealth W. According to the Markowitz definition we have that the following equality holds:

$$E[u(W + \tilde{Z})] = u[W - P(W, \tilde{Z})].$$

Consider the Taylor expansion[4] of the right hand side and of the left hand side of the above expression. Both $W + \tilde{Z}$ and $W - P(W, \tilde{Z})$ can been interpreted as deviations from the initial wealth W. Hence we shall compute the Taylor expansion of both sides of the equality for small deviations in a neighborhood of W.

1) Looking first at the right hand side obtain:

$$u(W - P(W, \tilde{Z})) = u(W) - P(W, \tilde{Z}) \times u'(W) + (\ldots),$$

where the term $(\ldots)$ denotes a negligible amount (more on this later);

2) For the left hand side instead one obtains:

$$E[u(W + \tilde{Z})] = E[u(W) + \tilde{Z}u'(W) + \frac{1}{2}\tilde{Z}^2 u''(W) + (\ldots)] =$$

$$u(W) + E(\tilde{Z})u'(W) + \frac{1}{2}E(\tilde{Z}^2)u''(W) + (\ldots)).$$

and since the lottery is actuarially neutral, $E(\tilde{Z}) = 0$, and setting the remaining terms in $(\ldots)$ to zero obtain:

$$u(W) + \frac{1}{2} \times E(\tilde{Z}^2)u''(W).$$

Now, notice that $E(\tilde{Z}^2)$ is the variance of $\tilde{Z}$ as by assumption $\tilde{Z}$ has mean zero. In order to emphasize that this is the variance let's write it as σ^2_Z. Also, following Pratt (1964), we consider only the first two terms of the expansion as the remaining part is very small (but not necessarily zero as we will see).
Equating the terms and solving for $P(W, \tilde{Z})$ we obtain the Arrow-Pratt measure of (local) **risk premium**:

$$P(W, \tilde{Z}) \simeq \frac{1}{2} \times \sigma^2_Z \times \left(-\frac{u''(W)}{u'(W)}\right)$$

Notice that the derivation is an approximation and holds for small bets only.[5]

[4] The Taylor series of the infinitely differentiable real function $f(x)$ at the real number a is the power series $f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$,

[5] An obvious question is under which conditions the derivation is exact, *i.e.*, under which conditions the residual terms in $(\ldots)$ are actually zero. It turns out that this is the case for CARA and normally distributed returns. For a proof see Eeckhaudt, Gollier and Schelinger, *Economic and Financial Decision under Risk,* Princeton University Press, 2005.

The ratio of the second to the first derivative in the last equation is the measure of the agent's risk aversion at the wealth level W, we call it **Absolute Risk Aversion** (ARA) and define it as follows:

Definition 2.6.1. *Absolute Risk Aversion:*

$$ARA \cong -\frac{u''(W)}{u'(W)}$$

Intuition: The first derivative is a measure of the marginal utility when the wealth increases, and this has a positive value. The second derivative is a measure of the change of the marginal utility and has a negative sign if the utility is concave. Their ratio tells us the *rate* of change (a fall if the function u is concave) of the marginal utility when wealth changes.
Notice that the value of the ARA is sensitive to the unit of account of wealth. So, the ARA will be different depending if we use gold or, say, pounds to measure individual wealth. This is obviously undesirable.

The measure of **Relative Risk Aversion** solves this problem by looking at the ratio of relative changes of the marginal utility and wealth:

Definition 2.6.2.

$$RRA \cong -\frac{du'(W)/u'(W)}{dW/W} = W \times (-\frac{u''(W)}{u'(W)}) = W \times ARA$$

Remark: ARA and RRA will be lower the higher the utility increment for a given increase in wealth (so a gamble with a positive expected payoff will be more attractive if the first derivative is high).

It is reasonable to expect that the ARA of an individual to decrease as her wealth increases. But this is not always true. The quadratic utility case is again an exception as the following exercise asks you to verify.

Example 2. *: Suppose that the utility function of an individual is given by $U(W) = aW - bW^2$. Computing the first and second derivative obtain $U'(W) = a - 2bW$ and $U''(W) = -2b$.*

Computing the ARA and RRA for this agent we obtain:

$$ARA = \frac{2b}{a - 2bW};$$

$$RRA = W\frac{2b}{a - 2bW}.$$

Notice that ARA increases with W. Why this property is not desirable as a realistic description of individual preferences?

Lecture 3

ASSETS, PORTFOLIOS AND ARBITRAGE[1]

3.1 Some definitions and results

Consider a simple economy lasting for two periods only, denoted by $t = 0, 1$, under uncertainty. Uncertainty resolves in the second period with two possible and mutually exclusive events. We call these events *states of the world* or *contingencies*, and denote them by $s = 1, 2$.

Suppose now that the agents living in this economy exchange contracts, that we call *securities* or, more generally, *financial assets* and denote them by a. These securities entitle the holder to receive a certain amount of consumption good if a certain contingency occurs. We denote the amount delivered in state s by security a by r_s^a and call this amount the asset's payoff in state s. Summarizing:

Definition 3.1.1. *: A* security *or* financial asset*, denoted by a, is a contract that promises to deliver an amount of future consumption, versus a payment today. The amount that the asset promises to deliver in state s is its* payoff *or* return*, say r_s^a and the payment today is the* price *of the asset, that we will denote by q_a.*

Definition 3.1.2. *We say that the asset payoff is* state contingent *if the payoff depends upon the realizations of uncertainty:* i.e., *r_1^a in state 1, r_2^a in state 2, with $r_1^a \neq r_2^a$. Otherwise, we say that the payoff is* state independent*.*

Suppose that in our simple economy there are two assets only, call them asset 1 and asset 2. Hence, $a = 1, 2$. The return of each asset $a = 1, 2$ is the vector $r^a = (r_1^a, r_2^a)$.

It will be useful to arrange the payoffs of the two assets in a matrix as follows:

$$\begin{bmatrix} r_1^1 & r_1^2 \\ r_2^1 & r_2^2 \end{bmatrix}$$

Let us denote by y_a^h the amount of assets a held by individual h. The collection of his assets will be his *portfolio*. In the case there are two assets, the portfolio of individual h is $y^h = (y_1^h, y_2^h)$. When the reference to the individual holder is irrelevant, we drop the superscript h.

Definition 3.1.3. *A* portfolio*, or* trading strategy*, is a collection of assets $y = (y_a : a = 1, 2, \dots)$.*

[1]In some parts of this lecture I follow and simplify the contents of Sec. 1.2 in Pliska, *Introduction to mathematical economics*, Blackwell (1998) and Sec of 2.2 in LeRoy and Werner, *Principles of financial economics*, Cambridge University Press (2001).

The *price of a portfolio* is given by the value of the assets it contains. E.g., the price of a portfolio containing two assets only, $y = (y_1, y_2)$, is given by $q_1 y_1 + q_2 y_2$.

The return of a portfolio is equal to the sum of the returns of the assets it contains. E.g., the return of a portfolio containing two assets, $a = 1, 2$, with returns of each asset a given by $r^a = (r_1^a, r_2^a)$ is $(r_1^1 y_1 + r_1^2 y_2,\, r_2^1 y_1 + r_2^2 y_2)$ or:

$$\begin{bmatrix} r_1^1 & r_1^2 \\ r_2^1 & r_2^2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

Definition 3.1.4. *A vector of returns is said to be* attainable *if there exists a portfolio that delivers those returns.*

The following facts follows from basic linear algebra results.

Proposition 3.1.1. *If the matrix of returns is full row rank then any vector of returns is attainable. If the matrix is full rank then each return is attanable by one and only one portfolio. If the matrix of returns fails to be full row rank, give returns are either not be attainable or attainable by multiple portfolio.*

Exercise 3.1.1. *Consider the following returns' matrix:*

$$\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$$

The price of the first asset and second asset are 1 and 3, respectively. Find the portfolio that delivers returns (5,3) and its price.

Exercise 3.1.2. *Consider the following returns' matrix:*

$$\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$$

Show that the matrix is not invertible. Also, show that the returns $(1, 2)$ *is not attainable by any portfolio but many portfolios (a continuum of them) could deliver returns* $(3, 3)$.

Definition 3.1.5. *If the matrix of assets' returns is full row rank, then the asset market is said to be* complete *otherwise the market is said to be* incomplete.

Exercise 3.1.3. *Suppose you are trading in two assets, with the general asset returns' matrix:*

$$\begin{bmatrix} r_1^1 & r_1^2 \\ r_2^1 & r_2^2 \end{bmatrix}$$

and general prices: q_1, q_2. *Find the two portfolios delivering returns* $(1, 0)$ *and* $(0, 1)$. *Also, price those portfolios*

3.2 Arrow securities

The securities obtained in Exercise 3.1.3 play an important role in the theory of finance. The security with payoff $(1, 0)$ pays only if state 1 occurs and the security with payoff $(0, 1)$ pays only if state 2 occurs. Notice that each of them delivers one unit of consumption good in one and only one state of the world and nothing otherwise. For this reason they are called *elementary securities* or *Arrow securities* from the name of the economist that first developed the theory. Summarizing:

Definition 3.2.1. *An asset delivering one unit of the consumption good in one state and zero otherwise is called an* elementary security *or an* Arrow security*.*

If there exists a complete set of Arrow securities (i.e. the number of independent Arrow securities is equal to the number of states) then the price of any asset a with return r_s^a in state s can be simply derived as follows:

$$q_a = \sum_{s=1}^{S} q_s^A r_s^a,$$

where q_s^A denotes the price of the Arrow security paying 1 unit of consumption good in state s. In fact, if a complete set of Arrow securities is available, any security with payoff r_s^a, $s = 1, \ldots S$ is equivalent from an investor point of view, to a portfolio containing $r_s^a, s = 1, \ldots S$ units of each Arrow security paying in state $s = 1, \ldots, S$. But then the security paying returns r_s^a and the portfolio of Arrow securities delivering exactly the same returns, must have the same price (*what if not?*). This is a no-arbitrage argument that we will develop later in this lecture.

Suppose now that you want to evaluate how much a risk neutral investor is willing to pay for a certain asset, say asset a, knowing its returns in each state r_s^a, the probability in each state π_s and the investor's time discount factor, say β. We know that a risk neutral agent will look only at the present value of the cash flow and hence he would be willing to pay an amount equal to:

$$q_a = \beta E r^a = \beta \sum_s \pi_s r_s^a.$$

There is a useful parallel between this formula and the pricing of an asset by elementary securities that can be constructed as follows. Consider the returns of a risk-free bond in a S-states economy. These can be arranged in a S-dimensional column vector as the following: $(1, \ldots, 1)'$ (the symbol " $'$ " denotes that the vector is transposed). Suppose now that a full set of Arrow securities is available and their prices, denoted by $q_s^A : s = 1, 2, \ldots, A$ are known. We can easily price the risk-less bond (denoted by q_f as follows:

$$q_f = \Sigma_{s=1}^S q_s^A.$$

The price q_f of the risk-less bond is the amount of first period good the market is willing to pay for a contract that promises one unit of good tomorrow, irrespective of the realization of uncertainty. This amount represents the value in terms of today consumption of one unit of consumption tomorrow. But this is equivalent to β, the time discount factor. We can then set $\beta = q_f$. Notice now that we can rewrite the pricing formula for the valuation of a generic asset a with returns r_s^a as follows:

$$q_a = \beta \Sigma_{s=1}^S \frac{q_s^A}{\Sigma_{s=1}^S q_s^A} r_s^a.$$

Notice also that $\Sigma_{s=1}^S \frac{q_s^A}{\Sigma_{s=1}^S q_s^A} = 1$ and that each term $\frac{q_s^A}{\Sigma_{s=1}^S q_s^A}$ premultiplying the return of the asset in state s plays the same role of the probability of state s in the valuation of the asset by a risk neutral agent with time discount factor β. For this reason, the terms $\frac{q_s^A}{\Sigma_{s=1}^S q_s^A}$ are called *risk neutral probabilities,* i.e. the probabilities that a risk neutral agent (the market?) with discount factor β would use to price/evaluate assets. Mind you that these are not the "real" or the "true" probabilities, but they play the role of probabilities in the risk neutral valuation for this economy.

3.3 Dominant portfolios

Definition 3.3.1. *A portfolio, call it $\hat{y} = (\hat{y}_1, \hat{y}_2)$, is said to dominate another portfolio, call it $y' = (y_1^{'}, y_2^{'})$, if the prices are the same and portfolio $\hat{y}$ delivers a strictly higher return in each state of the world than portfolio y', i.e.:*

$$(\hat{y}_1 r^1 + \hat{y}_2 r^2) \gg y_1^{'} r^1 + y_2^{'} r^2.$$

Proposition 3.3.1. *: If there exists a dominant portfolio then there exists a portfolio with zero price and strictly positive payoff.*

Suppose $\hat{y}$ dominates y'. In order to prove the proposition it suffices to consider the portfolio $y = \hat{y} - y'$. If there exists a portfolio with zero price and strictly positive returns, then this dominates the portfolio with zero price and zero returns in all states of the world.

3.4 The Law of One Price

The *Law of One Price* states that portfolios with the same returns must have the same price.

Proposition 3.4.1. *If there are no dominant portfolios, then the Law of One Price holds. The converse is not necessary true.*

Proposition 3.4.2. *The Law of One price holds if and only if any zero payoff portfolio has a zero price.*

In fact, suppose there is a portfolio with zero payoff and positive price. Then any zero payoff portfolio can be purchased at any price.

3.5 Arbitrage

Definition 3.5.1. *There is arbitrage opportunity if there exists a zero price portfolio delivering a non-negative return in all future states of the world with a strictly positive return in at least some state.*

Proposition 3.5.1. *If there exists a dominant portfolio, then there exists an arbitrage portfolio, but the converse is not necessary true.*

Concluding: the set of non-arbitrage prices/returns is strictly contained in the set of no dominant portfolios that is strictly contained in the set of price/returns satisfying the Law of One Price.

Lecture 4

FINANCIAL MARKETS

4.1 Introduction

In Lecture 1 we considered the problem of an individual having to choose her consumption stream overtime subject to her resources *i.e.* maximize her intertemporal preferences given the budget constraint. Also, we assumed that the agent had access to some sort of elementary financial opportunity: she could lend/borrow a certain amount of the consumption good and receive/give back that amount in the period after with interest. In Lecture 2 we introduced uncertainty and studied agents' attitude towards risk. We saw that risk averse agents (with preferences represented by a concave utility function) aim at smoothing consumption across contingencies.
The objective of this lecture is to show *how* financial markets actually allow agents to attain this objective. In particular, in this lecture we will consider financial markets that allow agents to perfectly hedge fluctuations in their income (or endowment). In the next lecture we will analyze the consequences of financial markets imperfections (*e.g.,* markets not sufficiently developed) and where perfect smoothing is not possible.
We will see that the fundamental property derives from the matrix of assets' payoffs and we will refer to this as the case of complete markets.
In particular, we want to show the following properties of financial markets:
1) financial markets allow agents to transfer income across time and across contingencies.
2) if the markets are "perfect" (*i.e.,* complete), the equilibrium allocation resulting from the trade in assets is optimal in the sense of Pareto.
3) if markets are "perfect," then assets returns do not affect the equilibrium allocation.
These basic but very important properties of financial markets, can be easily illustrated by a simple example.

4.2 Description of a simple economy

We consider an economy that extends over two periods $t = 0, 1$, under uncertainty. In the second period there are two possible realizations of uncertainty denoted by $s = 1, 2$. These states are mutually exclusive and we assume they are equally likely (so, the probability of each state is 1/2).
We suppose that there are two individuals in the economy, denoted by $h = 1, 2$. Each individual has to decide how much to consume in the two periods and in the two states.

Commodities

Suppose that there is only one good (say gold) in the first period, but there are 2 goods (say, gold and silver) in the second period. In Lecture 1 we specified that goods available at different points in time had to be considered as different commodities (gold today is a different commodity than gold tomorrow), even if they had the same intrinsic characteristics. Now, one further specification is necessary: goods available under different realizations of uncertainty have to be considered different commodities. Hence, if there are only two possible states of the world tomorrow, say rain and sunshine, and only one good is consumed, say gold, then *gold tomorrow if rain is the state of the world* is a different commodity than *gold tomorrow if sunshine is the state of the world.* So, in this economy -with 1 good at $t = 0$ and 2 goods per state at $t = 1$- there are 5 commodities in total (*can you give a complete list of them?*). Denote by x_0^h Mr. h demand of the commodity in the first period, by $x_{s,c}^h$ Mr. h consumption of good c in state s (*e.g.,* $x_{2,1}^2$ denotes Mr. 2 consumption of good 1 in state 2).

Endowments

We denote by e_0^h Mr. h endowment of the commodity in the first period and by $e_{s,c}^h$ Mr. h endowment of good c in state s. Suppose now that the two agents have the same endowment in the first period, given by 10 units of the good (so, $e_0^1 = e_0^2 = 10$). Also suppose that Mr. 1 has 10 units of each good in the second period and in the first state, but has nothing in the second state. To the contrary, Mr. 2 owns nothing in the first state, but has 20 units of each good in the second state. Notice that, given the initial allocation of endowments, agents in this economy face both *idiosyncratic* (*i.e.,* agent-specific) risk and aggregate risk (why?).

Preferences

In order to simplify matters, we assume agents' preferences to be the same and they are represented by the following utility function:

$$x_0^h + \frac{1}{2}(\ln x_{1,1}^h + \ln x_{1,2}^h) + \frac{1}{2}(\ln x_{2,1}^h + \ln x_{2,2}^h)$$

Notice that we are assuming that preferences are time and state separable and convex (*why?*). Moreover, the utility function is linear in the first period consuption and hence the first period marginal utility is constant and equal to 1. The latter will prove convenient in computing the equilibrium.

Assets

Suppose now that agents can trade in assets.

Recall from Lecture 3 that we defined a **security** (or an **asset**) as *a contract that promises to deliver a certain amount of tomorrow consumption, versus a payment today.* The total amount that the asset promises to deliver is its payoff (say, r) and the payment today is the price of the asset (say, q). We say that the asset payoff is state contingent if r depends upon the possible realizations of uncertainty: say, r_1 in state 1, r_2 in state 2.
Assume that two assets are traded in the asset market. We denoted each asset by $a = 1, 2$. The assets promise to deliver a certain amount of good 1 in each state. So r_s^a denotes the amount of the first good in state $s = 1, 2$ delivered by asset a. The price of the asset a is q_a, *i.e.,* q_1 is the price of asset 1 and q_2 is the price of asset 2.
As in Lecture 3, an asset a is characterized by its payoffs (r_s^a, $s = 1, 2$). The asset market is characterized by its payoff matrix:

$$\begin{bmatrix} r_1^1 & r_1^2 \\ r_2^1 & r_2^2 \end{bmatrix}$$

We start by analyzing the simpler case of Arrow (*elementary*) securities. In the next section we will analyze the more general case. The striking fact will be that thanks to financial market completeness the two cases deliver exactly the same equilibrium outcome.

4.3 Arrow securities

Assume now that agents trade in two securities that have the following particular payoff structure. The first security promises, to its holder, 1 unit of good 1 if state 1 occurs and nothing otherwise. Similarly, the second security promises, to its holder, 1 unit of the first good if state 2 occurs and nothing otherwise. The payoff matrix becomes:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Recall from Lecture 3 that assets of this type are called Arrow securities.[1]
Denote by y_1^h and y_2^h the amount of asset 1 and 2 that the individual h buys (or sells, if negative). The vector $y^h = (y_1^h, y_2^h)$ represents the portfolio of individual h.

4.4 Expectations

In the computation of the equilibrium of the economy, we are assuming that the agents can predict the future equilibrium prices. This implies that not only they know the possible future realization of their wealth and future preferences, but also know the realization of the wealth of the other agents too as well as their preferences and how this will affect prices. Basically, they know the *structure of the economy.* This is a major departure from the standard assumption of static economies, where agents only needed to know their own endowment and preferences. This assumption is however necessary as otherwise agents would not know how to anticipate future prices. This is essentially what is called the Rational Expectations Hypothesis.

Assumption 4.1. *Agents know the structure of the economy,* i.e., *they know preferences and endowments and how they determine prices.*

Under Rational Expectations, present and expected prices clear present and future markets.

4.5 Computing the equilibrium

We can now proceed to compute the **equilibrium** in this economy. We follow three steps:

Step 1) Given the prices, compute each agent's demand functions for all goods;
Step 2) Impose the equality of demand and supply in all markets (*i.e.,* impose the equilibrium condition);
Step 3) From the equilibrium conditions derive equilibrium prices and allocation.

Step 1. Compute each agent's demand function.

[1] Arrow (1964).

Let us write the budget constraints of individual 1 when he can buy and sell the securities. The individual has to satisfy three budget constraints, one for each spot: the first budget constraint is relative to the first period consumption, the second and third budget constraints are relative to second period consumption in the first and second state, respectively. Now in the first period each agent has to decide his optimal portfolio, *i.e.,* the amount of each asset to buy or sell. Also assume that the first good is the numeraire at each time and state. Let us denote with $\tilde{p}_{1,2}$ and $\tilde{p}_{2,2}$ the prices of commodity (1,2) and (2,2), respectively (the tilde is to emphasize that we are talking about expected prices). The difference is not just in notation. Notice that now the prices are *expected* prices, *i.e.,* agents have to form expectations on the future prices of goods. We are implicitly assuming that traders hold the correct price expectations.

Mr. 1 budget constraint is given by:

$$\begin{aligned} & x_0^1 + q_1 y_1^1 + q_2 y_2^1 = 10 && \text{(first period)} \\ & x_{1,1}^1 + \tilde{p}_{1,2} x_{1,2}^1 = 10 + 10\tilde{p}_{1,2} + y_1^1; && \text{(second period, state 1)} \\ & x_{2,1}^1 + \tilde{p}_{2,2} x_{2,2}^1 = y_2^1; && \text{(second period, state 2).} \end{aligned}$$

Exercise 4.5.1. *Write agent 1's budget constraint in matrix form.*

Rearranging the last two constraints obtain:

$$y_1^1 = x_{1,1}^1 + \tilde{p}_{1,2} x_{1,2}^1 - 10 - 10\tilde{p}_{1,2};$$

$$y_2^1 = x_{2,1}^1 + \tilde{p}_{2,2} x_{2,2}^1.$$

Substituting the values of y_1^1 and y_2^1 in the first period budget constraint and re-arranging terms, we obtain:

$$x_0^1 + q_1 x_{1,1}^1 + q_1 \tilde{p}_{1,2} x_{1,2}^1 + q_2 x_{2,1}^1 + q_2 \tilde{p}_{2,2} x_{2,2}^1 = 10 + 10q_1 + 10q_1\tilde{p}_{1,2}.$$

We can now solve Mr. 1 maximization problem. This is given by:

$$\begin{aligned} \max \quad & x_0^1 + \tfrac{1}{2}(\ln x_{1,1}^1 + \ln x_{1,2}^1) + \tfrac{1}{2}(\ln x_{2,1}^1 + \ln x_{2,2}^1) \\ \text{s.t.} \quad & x_0^1 + q_1 x_{1,1}^1 + q_1 \tilde{p}_{1,2} x_{1,2}^1 + q_2 x_{2,1}^1 + q_2 \tilde{p}_{2,2} x_{2,2}^1 = 10 + 10q_1 + 10q_1\tilde{p}_{1,2}. \end{aligned}$$

Solving x_0^1 from the constraint above, replacing it into the utility function and computing the first order conditions for the problem obtain:

$$\begin{aligned} x_{1,1}^1 : \quad & -q_1 + \tfrac{1}{2x_{1,1}^1} = 0 && \Rightarrow \quad x_{1,1}^1 = \tfrac{1}{2q_1}; \\ x_{1,2}^1 : \quad & -q_1\tilde{p}_{1,2} + \tfrac{1}{2x_{1,2}^1} = 0 && \Rightarrow \quad x_{1,2}^1 = \tfrac{1}{2q_1\tilde{p}_{1,2}}; \\ x_{2,1}^1 : \quad & -q_2 + \tfrac{1}{2x_{2,1}^1} = 0 && \Rightarrow \quad x_{2,1}^1 = \tfrac{1}{2q_2}; \\ x_{2,2}^1 : \quad & -q_2\tilde{p}_{2,2} + \tfrac{1}{2x_{2,2}^1} = 0 && \Rightarrow \quad x_{2,2}^1 = \tfrac{1}{2q_2\tilde{p}_{2,2}}. \end{aligned}$$

On the right hand side we have the demand functions of agent 1.[2]

[2]Here we are abusing notation as we are not distinguishing between control variables, functions and actual values. To be more precise, we should write the demand functions as $x_{s,c}^h(\mathbf{p})$ where $\mathbf{p}$ is the vector of prices.

Similarly, for the second individual (remember that he has 20 units of each good in state 2 but nothing in state 2) you can easily show that the budget constraint reduces to:

$$x_0^1 + q_1 x_{1,1} + q_1 \tilde{p}_{1,2} x_{1,2}^1 + q_2 x_{2,1}^1 + q_2 \tilde{p}_{2,2} x_{2,2}^1 = 10 + 20q_2 + 20q_2\tilde{p}_{2,2}$$

Following the same procedure, you can show that Mr. 2 demand functions are given by:

$$x_{1,1}^2 = \frac{1}{2q_1};$$
$$x_{1,2}^2 = \frac{1}{2q_1\tilde{p}_{1,2}};$$
$$x_{2,1}^2 = \frac{1}{2q_2};$$
$$x_{2,2}^2 = \frac{1}{2q_2\tilde{p}_{2,2}}.$$

Step 2. Impose the market clearing conditions and obtain equilibrium prices.

Now, we can compute the equilibrium prices and allocation. We impose that the total demand is equal to the total supply for each good and derive the price implied by this condition (equivalently: prices adjust to equate demand and supply). The total supply of a certain good (right hand side of the equality) is given by the sum of agents endowments of that good: notice that in this economy, in the second period there are 10 units of each good in total if state 1 occurs and 20 if state 2 occurs. The total demand (left hand side of the equality) comes from the agents' demand that we just computed. These conditions are called the **market clearing conditions** (*m.c.c.*). Here are the m.c.c. for the second period goods:

$$x_{1,1}^1 + x_{1,1}^2 = 10 \quad \Rightarrow \quad \frac{2}{2q_1} = 10;$$
$$x_{1,2}^1 + x_{1,2}^2 = 10 \quad \Rightarrow \quad \frac{2}{2q_1\tilde{p}_{1,2}} = 10;$$
$$x_{2,1}^1 + x_{2,1}^2 = 20 \quad \Rightarrow \quad \frac{2}{2q_2} = 20;$$
$$x_{2,2}^1 + x_{2,2}^2 = 20 \quad \Rightarrow \quad \frac{2}{2q_2\tilde{p}_{2,2}} = 20.$$

From the m.c.c. we can derive the equilibrium prices for the second period goods. Notice that we identify equilibrium values with an asterisk:

$$q_1^* = \frac{1}{10}; \quad \tilde{p}_{1,2}^* = 1;$$
$$q_2^* = \frac{1}{20}; \quad \tilde{p}_{2,2}^* = 1.$$

Step 3. Derive the equilibrium allocation.

We can now substitute these values into each agent's demand function and obtain for each agent $h = 1, 2$ obtain:

$$x^{h*}_{1,1} = 5; .$$
$$x^{h*}_{1,2} = 5;$$
$$x^{h*}_{2,1} = 10;$$
$$x^{h*}_{2,2} = 10.$$

This is the equilibrium allocation in the second period commodity markets, *i.e.,* this is what each agent consumes after the exchange has taken place.
By substituting these equilibrium values in the second period budget constraints you can find that the equilibrium portfolios are given by:

$$y^{1*}_1 = -10 = -y^{2*}_1, \text{and } y^{1*}_2 = 20 = -y^{2*}_2.$$

Since these are porfolios composed of Arrow securities it is immediate to see that the first agent is "moving" 10 units of wealth out of state 1 in order to "move" 20 units of wealth into state 2. Viceversa for agent 2.
In order to find the first period allocation (x^{h*}_0) we just need to substitute the equilibrium values just found in the budget constraint of the individuals.

Exercise 4.5.2. *Define a financial equilibrium for this economy.*

Several remarks are in order. The following exercises ask you to think about the most important ones.

Exercise 4.5.3. *Compute the risk neutral probabilities for this economy. Are they the same as the actual probabilities? Compute the discount factor.*

Exercise 4.5.4. *In the example above, the price of Arrow security 2 is lower than the price Arrow security 1 but the commodities have the same prices. Explain why.*

Exercise 4.5.5. *Mr. 2 owns twice as much in state 2 than Mr. 1 does in state 1. However, at equilibrium they end up consuming the same amount. Explain why.*

Exercise 4.5.6. *In the example, given their preferences, agents would like to perfectly smooth consumption across states. However, in the final allocation of resources, they consume the double in state 2 with respect to state 1. Hence, in this economy full insurance for all agents is not possible. Explain why.*

Here is one of the fundamental results of the lecture:

Claim 4.5.1. *The equilibrium of an economy with a complete set of Arrow securities is Pareto optimal.*

The proof of the claim 4.5.1 is left as an exercise (Hint: Show that the equilibrium marginal rates of substitution relative to any two goods are the same across agents.).

Exercise 4.5.7. ***A bit more advanced****. Show that, in the above example, prices of Arrow securities equal agents' Lagrangean multipliers (*i.e., *the price of each Arrow security is equal to the marginal utility of income).*

Exercise 4.5.8. *Using the solution to Exercise 4.5.7 show that the price of each asset is equal to the weighted average of its returns, where the weight are the prices of the Arrow securities.*

4.6 The case with general returns

In this section we show that Claim 4.5.1 holds in a much more general set up and for any assets' return provided markets are complete. There is more than one way to show this in our example. One possibility is to redo all the above computations. This is tedious. A second possibility is to show that the agent budget constraint, when re-arranged and assuming the Law of One Price holds, is the same budget constraint that the individual faces at equilibrium prices in the case of trade of Arrow securities. Hence the two situations must deliver the same demand functions and the same equilibrium allocation.
Suppose now that the agents trade in a complete set of assets with returns described by the matrix:

$$\begin{bmatrix} r_1^1 & r_1^2 \\ r_2^1 & r_2^2 \end{bmatrix}$$

Again, we assume that the matrix has full rank (i.e: the matrix can be inverted).

Let $\tilde{q}_a$ be the price of asset a (We use the tilde in order to distinguish it from the prices of the Arrow securities).

We want to show that the trade in a complete set of assets leads to exactly the same equilibrium allocation occurring under the case of Arrow securities.

Re-write Mr. 1 budget constraint and obtain:

$$x_0 + \tilde{q}_1 y_1 + \tilde{q}_2 y_2 = 10$$

$$x_{1,1} + \tilde{p}_{1,2} x_{1,2} = 10 + 10\tilde{p}_{1,2} + r_1^1 y_1 + r_1^2 y_2$$

$$x_{2,1} + \tilde{p}_{2,2} x_{2,2} = r_2^1 y_1 + r_2^2 y_2.$$

Writing the second period budget constraints in matrix form obtain:

$$\begin{bmatrix} x_{1,1} + \tilde{p}_{1,2} x_{1,2} - 10 - 10\tilde{p}_{1,2} \\ x_{2,1} + \tilde{p}_{2,2} x_{2,2} \end{bmatrix} = \begin{bmatrix} r_1^1 & r_1^2 \\ r_2^1 & r_2^2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

By market completeness we can invert the matrix and solve for the portfolio:

$$\begin{bmatrix} r_1^1 & r_1^2 \\ r_2^1 & r_2^2 \end{bmatrix}^{-1} \begin{bmatrix} x_{1,1} + \tilde{p}_{1,2} x_{1,2} - 10 - 10\tilde{p}_{1,2} \\ x_{2,1} + \tilde{p}_{2,2} x_{2,2} \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

equivalently:

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \frac{1}{r_1^1 r_2^2 - r_1^2 r_2^1} \begin{bmatrix} r_2^2 & -r_1^2 \\ -r_2^1 & r_1^1 \end{bmatrix} \begin{bmatrix} x_{1,1} + \tilde{p}_{1,2} x_{1,2} - 10 - 10\tilde{p}_{1,2} \\ x_{2,1} + \tilde{p}_{2,2} x_{2,2} \end{bmatrix}.$$

i.e.,

$y_1 = \frac{1}{r_1^1 r_2^2 - r_1^2 r_2^1}[r_2^2(x_{1,1} + \tilde{p}_{1,2}x_{1,2} - 10 - 10\tilde{p}_{1,2}) - r_1^2(x_{2,1} + \tilde{p}_{2,2}x_{2,2})]$,

$y_2 = \frac{1}{r_1^1 r_2^2 - r_1^2 r_2^1}[-r_2^1(x_{1,1} + \tilde{p}_{1,2}x_{1,2} - 10 - 10\tilde{p}_{1,2}) + r_1^1(x_{2,1} + \tilde{p}_{2,2}x_{2,2})]$.

Remark: in this simple two-states example, you do not need to use the matrix formulation to obtain the portfolio of the individuals! Here I am using it just to make you remark where the condition on the invertibility of the matrix comes in.

Substituting back in the first period budget constraint and rearranging terms we obtain:

$$x_0 + \frac{1}{r_1^1 r_2^2 - r_1^2 r_2^1}(r_2^2\tilde{q}_1 - r_2^1\tilde{q}_2)(x_{1,1} + \tilde{p}_{1,2}x_{1,2} - 10 - 10\tilde{p}_{1,2}) +$$
$$\frac{1}{r_1^1 r_2^2 - r_1^2 r_2^1}(r_1^1\tilde{q}_2 - r_1^2\tilde{q}_1)(x_{2,1} + \tilde{p}_{2,2}x_{2,2}) = 10$$

Similarly for the second consumer.

Up to here we have simply re-arranged the three constraints by reducing them to a unique one. The invertibility of the matrix of assets' returns has proved to be crucial.

Now, we simply have to recall from Lecture 3 that these assets are combinations of Arrow securities. In fact, we can construct the first assets by buying r_1^1 units of the first Arrow security and r_2^1 units of the second Arrow security. Similarly for asset 2. Hence, at equilibrium, the prices of these securities must be given by:

$$\tilde{q}_1^* = q_1^* r_1^1 + q_2^* r_2^1$$

$$\tilde{q}_2^* = q_1^* r_1^2 + q_2^* r_2^2$$

(remember that the starred prices are the equilibrium prices computed in the Arrow securities economy).

If you substitute these prices in the budget constraint above you obtain:

$$x_0^1 + q_1^*(x_{1,1} + \tilde{p}_{1,2}x_{1,2} - 10 - 10\tilde{p}_{1,2}) + q_2^*(x_{2,1} + \tilde{p}_{2,2}x_{2,2}) = 10.$$

This is budget constraint of the economy with Arrow securities! This implies that the allocation in the economy with general returns is the same allocation obtained when Arrow securities are traded.

It follows that when the financial markets are complete, agents are able to perfectly trade commodities across contingencies. The equilibrium arising in these economies is Pareto optimal.

Exercise 4.6.1. ***More advanced.*** *Use the Lagrange method to solve the more general problem and show that the price of each asset $a = 1, 2$ is given by:*

$$q_a = \sum_s \frac{\mu_s}{\mu_0} r_s^a.$$

where μ_0 and μ_s are the Lagrangean multiplier in $t = 0$ and state s, respectively.

Exercise 4.6.2. *If markets are complete, individual consumption is perfectly correlated to aggregate consumption (and not individual wealth). Explain.*

Lecture 5

MARKET INCOMPLETENESS AND THE ROLE OF OPTIONS

Until now we have studied the properties of economies with complete financial markets. The aim of this lecture is to analyze the implications of market incompleteness. We will also briefly introduce options, a type of derivative contract and look at their role in helping traders to better smooth consumption across states of the world.

5.1 Market incompleteness

Consider an economy extending over two periods, under uncertainty. As usual, uncertainty is represented by two possible realizations/states of the world, denoted $s = 1, 2$, equally probable. There are two agents, denoted by $h = 1, 2$. Only one good is consumed in each contingency. We denote by x_c^h the consumption of commodity $c = 0, 1, 2$ by agent h, where $c = 0$ denotes the consumption in the first period, $c = 1$ and $c = 2$ denotes the consumption in the second period in the first and second state, respectively. Agents preferences are the same and represented by the von Neumann-Morgestern utility function:

$$\frac{1}{2}x_0^h + E(x^h - \frac{1}{2}(x^h)^2).$$

equivalently, this can be written as (why?):

$$x_0^h + x_1^h - \frac{1}{2}(x_1^h)^2 + x_2^h - \frac{1}{2}(x_2^h)^2.$$

Agent 1 and 2's endowment is $e^1 = (1, 0.5, 0.4)$ and $e^2 = (1, 0.4, 0.5)$ respectively.

Let good 0 be the numeraire. In this economy there is obviously scope for trade. Agents are risk averse since second period preferences are strictly convex, have the same preferences but different endowments. Hence, they would probably like to transfer some consumption across states: agent 1 from the first to the second state and viceversa for agent 2. This is also perfectly compatible with the original distribution of resources. Hence, without "imperfections", we should obtain that agents do perform these desirable trades. You are asked to verify this in the following exercise.

Exercise 5.1.1. *Assume that agents trade two Arrow securities. Define an equilibrium for this economy. Show that the equilibrium is Pareto optimal and agents can perfectly smooth consumption across states.*

Assume now that the asset market is incomplete: only one asset is available for transferring consumption across time and the two states of the world. The price of the asset is q and the assets returns one unit of consumption in period $t = 1$ independently of the realization of the state (so $r = (1,1)$). Let y^h denotes agent $h = 1,2$ portfolio. The optimal consumption/portfolio choice for agent 1 is the solution to the maximization problem:

$$\max \quad x_0^1 + x_1^1 - \frac{1}{2}(x_1^1)^2 + x_2^1 - \frac{1}{2}(x_2^1)^2,$$
$$\text{s.t.} \quad x_0 + qy^1 = 1;$$
$$x_1^1 = 0.5 + y^1;$$
$$x_2^1 = 0.4 + y^1.$$

Exercise 5.1.2. *Define a competitive equilibrium for the economy with incomplete financial markets.*

It is easy to show (*do it!*) that the optimal portfolio choice is: $y^1 = \frac{1.1-q}{2}$ and this is equal to Agent 2 optimal portfolio choice, *i.e.,*

$$y^2 = \frac{1.1-q}{2}.$$

The asset market clearing condition requires that:

$$\frac{1.1-q}{2} + \frac{1.1-q}{2} = 0,$$

and hence the equilibrium price and portfolios are given by:

$$q^* = 1.1.$$

$$y^{1*} = y^{2*} = 0.$$

Exercise 5.1.3. *Show that the equilibrium for the economy with incomplete markets is the autarkic equilibrium.*

Notice that with one asset only paying (1,1), agents are not willing to trade in the asset market at all (the market is inactive as no trade takes place). The intuition for what is happening is simple: if he could, agent 1 would like to transfer some of his endowment from state 1 into state 2 (viceversa for agent 2). By buying the asset, he transfers endowment in *both* states; by selling, he transfers endowment out of *both* states. It is like he is buying second period goods in bundles, without the opportunity to disentangle them. Hence the unwillingness to trade. The equilibrium is obviously suboptimal, and you are asked to show that this is the case in the next exercise.

Exercise 5.1.4. *Show that the competitive equilibrium with incomplete asset markets is suboptimal.*

Exercise 5.1.5. *Suppose now that the asset pays* (r_1, r_2) *where* $r_1 \neq r_2$ *and agent 1's endowment is* $(1, 0.3, 0.4)$ *and agent 2's endowment is* $(1, 0.5, 0.2)$. *Will equilibrium allocation depend on the assets returns?*

5.2 Options

Definition 5.2.1. *A (European)* option *is an asset that allows the holder to buy at a fixed price (*the strike *price) another asset (the* primary asset*) after the state of the world is revealed but before the assets' returns are paid.*

An option is then just another type of asset. As any other asset, its price will be determined at equilibrium and it will obviously depend on its payoff. An investor will not exercise the option if the strike price is higher than the return. Hence, assuming there are S states of nature, if the primary asset has a vector of returns given by $r = (r_1, r_2, \ldots, r_s, \ldots, r_S)$ then an option with strike price c will have returns given by

$$r(c) = (\max\{0, r_1 - c\}, \max\{0, r_2 - c\}, \ldots, \max\{0, r_s\}, \ldots, \max\{0, r_S - c\}).$$

Now the option can be priced as any other asset. Suppose a complete set of Arrow securities is available. If the price of Arrow security s (the security that has a positive, unitary return in state s) is given by q_s^A, then the equilibrium price of the option is given by:

$$\sum_{s=1}^{S} q_s^A \max\{0, r_s - c\}.$$

An example can clarify. Suppose $S = 2$ (so, we have two states only here). An option is issued on a primary asset with returns $(1, 2)$ with strike price 1.5 (this is what the option holder has to pay in order to exercise the option). Then, the holder will exercise the option only in state 2, receiving a return of 0.5 giving a return vector of $(0, 0.5)$. If a complete set of Arrow securities is available with prices, say, $(1, 2)$ (*i.e.,* 1 for security 1 and 2 for security 2) , then the price of the option will be 1 (why?).

5.3 Spanning through options

If the asset market is complete, issuing an option will have no effect on the asset market (why?) as neither asset prices nor the equilibrium consumption will be affected (why?). However, portfolio holding will be indeterminate (why?), but this is inessential (why?).

This is not the case if the asset markets are incomplete. Options might actually have the important function of completing markets. Consider an economy extending over two periods with 4 states of possible realizations of uncertainty in the second period.[1] Suppose there is only one (primary) asset traded, with returns' vector given by $r = (4, 3, 2, 1)$. An option with strike price c would have a vector of returns

$$r(c) = (\max\{0, 4 - c\}, \max\{0, 3 - c\}, \max\{0, 2 - c\}, \max\{0, 1 - c\}.$$

Now, by changing the strike price c we can generate several new assets and actually complete the market. Consider the issue of 3 options (asset 2, 3 and 4 respectively) with strike price 3.5, 2.5, 1.5. This will do. The matrix of assets' returns will be have full rank:

$$\begin{bmatrix} 4 & .5 & 1.5 & 2.5 \\ 3 & 0 & 0.5 & 1.5 \\ 2 & 0 & 0 & 0.5 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

[1] Here I am following an example given in Mas-Colell, Whinston and Green (1994).

Lecture 6

INFORMATIONAL AND ECONOMIC EFFICIENCY

6.1 Introduction

When agents are asymmetrically informed, prices do not only convey information on the aggregate scarcity of resources but also convey private information across agents. Agents will observe prices and will try to refine their information with the information conveyed by prices. When prices do convey information across agents, we say that they are *informational efficient.* Many authors have investigated the informative contents of asset prices both theoretically and empirically. Fama (1976) proposed a classification of markets according to price informational efficiency, distinguishing three forms (weak, semi-strong and strong), a subject beyond the scope of this course.

In this lecture we will try to understand what informational efficiency means and under which conditions or assumptions it is satisfied. The second part of the lecture we will argue that an informational efficient price mechanism is a not necessary condition of economic efficiency.

6.2 Informationally efficient prices and the Efficient Market Hypothesis

In the following development of the theory, we go one step further from the Rational Expectations Hypothesis with another departure from the basic and minimal assumptions pf the static economies of the Walrasian Equilibrium. This is the basis from the *Efficient Markets Hypothesis.* This is one of the most controvertial hypothesis in finance.

Assumption 6.1. *Agents use their knowledge about price formation in order to refine their information with the information revealed by prices.*

In order to show how prices reflect information private across agents consider the following, two-agent economy, extending over two periods with two possible states in the second period, denoted by H and L. In the first period agents exchange one consumption good versus a bond promising one unit of the good in each state of the world. Denote the asset price, agents' endowments, portfolios and consumption as in the previous lectures, with the only change that now the subindex $s = H, L$.

Agents' preferences are identical and described by the utility function: $x_0^h + E[x^h - \frac{1}{2}(x^h)^2]$.

Agents are asymmetrically informed on the future realization of contingencies. These differences in information are reflected in the probabilities they attach to the realization of each state. In particular, let us assume that the first agent knows which state will occur in the second period (so he attaches probability 1 or 0 to either state, depending on his information) and that the second agent does not have this information and, to the best of his knowledge, assigns equal probability to each realization. Each agent $h = 1, 2$ is endowed with e_0^h units if the good in the first period and e_s^h units if the good in the second period if state $s = H, L$ occurs.

The equilibrium price of the bond is informative if it reflects the information of the first agent. At equilibrium, we should obtain two different prices, say q_H and q_L depending on whether the informed agent attaches probability 1 to state H or L.
In order to find the informative equilibrium prices let us assume that once this information is acquired by the uniformed by looking at the price of the bond, the agent re-adjusts his portfolio holding to reflect what he learns from the price. For didactict purposuse, we will assume that this occurs in stages but we should think about this as if it was occurring in no time or as if trade would occur only once the second agent has fully updated his choice.[1]
The role of the following assumptions will be clearer later:

Assumption 6.2. $e_H^1 \neq e_L^1$

Assumption 6.3. $\sum e_s^h = e_s^h$

Suppose that the state is H. Since agent 1 knows this is the case, he assigns probability 1 to state H and 0 to state L and solves the maximization problem:

$$\begin{aligned} \max \quad & x_0^1 + x_H^1 - \tfrac{1}{2}(x_H^1)^2 \\ s.t. \quad & x_0^1 + qy = e_0^1 \\ & x_H^1 = e_H^1 + y \end{aligned}$$

or:

$$\max e_0^1 - qy + (e_H^1 + y) - \frac{1}{2}(e_H^1 + y)^2$$

with first order conditions:

$$-q + 1 - e_H^1 - y = 0;$$

and optimal portfolio:

$$y^1 = 1 - e_H - q.$$

The second agent does attaches probability $\frac{1}{2}$ to each realization and uses them to compute the optimal portfolio holding by solving the following problem[2]:

$$\max e_0^2 - qy + E[(e^2 + y) - \frac{1}{2}(e^2 + y)^2],$$

with first order condition:

$$-q + E[1 - e^2 + y] = 0,$$

or:

$$-q + 1 - Ee^2 - y = 0.$$

[1] These are assumptions that attracted a number of criticisms to Rational Expections Hypothesis.
[2] The upper index 2 to the endowment e^2 denotes agent 2 and not square of e

Then:

$$y^2 = 1 - q - Ee^2.$$

At equilibrium,

$$y^1 + y^2 = 0.$$

Substituting for y^1 and y^2 obtain in the state H:

$$q = \frac{2 - e_H^1 - Ee^2}{2},$$

Agent 2 observes q and deducts that this can only arise if agent 1 knows that the state is H. In fact, if he knew that the state was L the price should have been:

$$q = \frac{2 - e_L^1 - Ee^2}{2},$$

Hence, agent 2 readjusts his portfolio with the new information it has acquired by setting $Ee^2 = e_H^2$ so that the market clears at:

$$q_H^* = \frac{2 - e_H^1 - e_H^2}{2},$$

Similarly, if agent 1 were informed that the state was L then the equilibrium price would have been:

$$q_L^* = \frac{2 - e_L^1 - e_L^2}{2},$$

Exercise 6.2.1. *What are the consequences for the computation above if Assumptions 6.2 and 6.3 not holding?*

Exercise 6.2.2. *What would happen if assumptions 6.2 and 6.3 do not hold?*

6.3 Economic efficiency vs. informational efficiency: the "Hirshleifer effect"

There is an extensive finance literature, both theoretical and empirical, analyzing the informational properties of asset prices. A question that has been addressed is whether more information is always necessary to attain economic efficiency, equivalently, whether there might be important instances where more information might actually lead to Pareto inefficient equilibria.

In 1971 Hirshleifer[3] pointed out that information can adversely affect the volume of traded assets and hence prevent agents from taking advantage of insurance opportunities available without information.

Here's a simple example that explains how this can happen and when it is the case.

Consider the following one good, two agents, two periods economy. In the second period there are two, equally likely, states of the world $s = 1, 2$ and endowments are $e_1^1 = e_2^2 = 1$ and $e_2^1 = e_1^2 = 0$ (hence, no aggregate risk). There is no consumption in the first period and the utility of each agent in each contingency is simply $\sqrt{x_s^h}$.

[3]Hirshleifer (1971), *The private and social value of information*, American Economic Review, 61, 561-574.

There is one asset trade in the first period with price q and payoff $(-\frac{1}{2}, \frac{1}{2})$.
Suppose first that neither agent "knows" with certainty the state of the world tomorrow. They trade in the asset (agent 1 buys one unit and agent 2 sells one unit of the asset) and both agents $h = 1, 2$ reach the following equilibrium allocation (show why) for all $s = 1, 2$:

$$x_s^h = \frac{1}{2},$$

hence, agents are able to perfectly insure versus future contingencies.

Their utility level is given by:

$$\frac{1}{2}\sqrt{0.5} + \frac{1}{2}\sqrt{0.5} = \sqrt{0.5} \approx 0.707.$$

Suppose now that a third informed agent offers to the two uninformed ones to publicly announce the future state of the world *before* agent 1 and 2 can trade in the asset. Are the two traders willing to accept this information?
The uninformed agents, before the informed one announces the real realization of the state, will attach probability $\frac{1}{2}$ that he might reveal state 1 or state 2. If the third agent announces state 1, agent 1 will have utility level 1 and agent 2 level 0. Vice versa if the state is 2. However, *before* this information is revealed, the agents compute their expected utility knowing that either state might be announced with probability $\frac{1}{2}$:

$$\frac{1}{2}\sqrt{1} + \frac{1}{2} \times 0 = 0.5 < 0.707.$$

Similarly, for agent 2:

$$\frac{1}{2} \times 0 + \frac{1}{2} \times \sqrt{1} = 0.5 < 0.707.$$

Both agents are worse off with information than without.

The information conveyed by the third agents has destroyed the insurance opportunities of the agents. In fact, after the announcement has been made, one of the agents (Mr. 1 if state 1 is announced and Mr. 2 otherwise) will not be willing to trade in the asset (why?).

Lecture 7

FROM GENERAL EQUILIBRIUM TO MEAN VARIANCE ANALYSIS

7.1 Introduction

The expected utility approach is quite powerful in predicting agents' optimal portfolio choice and asset pricing. Also, it can predict the welfare properties of the equilibrium allocations. Empirically, however, is more difficult to test.
As a consequence we have to turn toward a simpler description of agents' choices. We will assume that while selecting their optimal portfolios, agents care only about the mean and variance of the returns (hence the name: mean-variance approach). In particular, if they prefer more to less and dislike risk, they should like portfolios with higher expected returns and lower variance. The characterization of this method of selecting portfolios will allow us to derive a simple and widely used equlibrium formula of asset pricing, namely Capital Asset Pricing Model (CAPM). In order to proceed, we need to recall how to compute portfolio rates of returns' mean and variance and some of their properties. Then we want to show which assumptions on agents' utility functions or on the distribution of assets's rates of returns justify such an approach. This is relevant in order to appreciate the usefulness of this type of models but also to be aware of their weaknesses.

7.2 Definitions

In this section we recall the definitions and the most important properties of the expected value, the variance and the correlation coefficient. We then apply those concepts to portfolios' rates of returns. Notice that here I am abusing some of the notation. For instance, we denote by r the *rate* of return and not simply the return of a portfolio or asset as we did in the previous lectures.

7.2.1 Expected value

A simple way to think about assets' rates of return is to model them as random variables. Similarly, portfolio returns can be modeled as the weighted sum of random variables.
Consider an asset a promising rates of returns $r^a_s, s = 1, \ldots, N$ with probability $\pi_s, s = 1, \ldots, N$ (here N denotes the total number of contingencies).

Definition 7.2.1. *: The **expected value** of the rate of return of asset a is:*

$$Er^a = \sum_{n=1}^{N} \pi_n r^a_n.$$

Let D be any constant and denote by b as second asset keeping similar notation used for asset a. Recall the following properties of linearity of the expectation operator is linear:

1) $E(D) = D$;
2) $E(D + r^a) = D + Er^a$;
3) $E(r^a + r^b) = Er^a + Er^b$;
4) $E(D \times r^a) = DEr^a$.

Exercise 7.2.1. *: Prove the above properties of the expected value.*

7.2.2 Variance

We denote the variance of a random variable, say r^a (in our case the variance of the rate of returns of an asset), by σ_a^2 or by $Var(r^a)$.

Definition 7.2.2. *: The **variance** of the rates of return of asset a is given by:*

$$\sigma_a^2 = E[r^a - Er^a]^2 = \sum_n \pi_n (r_n^a - Er^a)^2$$

Remark that:

$$\sigma_a^2 = E[(r^a)^2 + (Er^a)^2 - 2r^a Er^a] = E[(r^a)^2] + (Er^a)^2 - 2(Er^a)^2 = E[(r^a)^2] - (Er^a)^2$$

Exercise 7.2.2. *: Show the following properties of the variance where C is a constant and r^a is a random variable (in our case the rate of return of asset a):*

$$1) Var(C) = 0;$$
$$2) Var(C + r^a) = Var(r^a);$$
$$3) Var(Cr^a) = C^2 Var(r^a).$$

Definition 7.2.3. *: The **standard deviation (s.d.)** is the square root of the variance. The s.d. of the returns of asset a is denoted by σ_a where $\sigma_a = \sqrt{\sigma_a^2}$.*

7.2.3 Portfolio mean and variance: two assets case

Suppose that a portfolio is composed by portions X_a and X_b of asset a and asset b, respectively. Let us compute the expected value and the variance of the returns of such a portfolio. Let r^p the rate of return of the portfolio (also a random variable).

Expected value

Using the properties of expectations we can easily compute the portfolio expected rate of return:

$$E(r^p) = E(X_a r^a + X_b r^b) = E(X_a r^a) + E(X_b r^b) = X_a Er^a + X_b Er^b$$

The first equality comes from property 3 and the second equality from property 4.

Variance

$$
\begin{aligned}
Var(r^p) &= E[r^p - Er^p]^2 = \\
&= E[(X_a r^a + X_b r^b) - (X_a Er^a + X_b Er^b)]^2 \\
&= E[X_a(r^a - Er^a) + X_b(r^b - Er^b)]^2 \\
&= E[X_a^2(r^a - Er^a)^2 + X_b^2(r^b - Er^b)^2 \\
&\quad +2X_aX_b(r^a - Er^a)(r^b - Er^b)] \\
&= X_a^2 Var(r^a) + X_b^2 Var(r^b) + 2X_aX_b Cov(r^a, r^b).
\end{aligned}
$$

where $Cov(r^a, r^b) = E[(r^a - Er^a)(r^b - Er^b)]$, defines the covariance of the returns of asset a and b.

Exercise 7.2.3. *Consider a two period, two-state economy. Suppose the rates of return of two assets, a and b, take values $r^a = (1.2, 2.4)$ and $r^b = (3.2, 0.4)$, respectively, with probabilities $\pi_1 = \pi_2$. Compute the variance of the portfolio when the share of asset a is given by: $X_a = 0$, $X_a = 1/4$, $X_a = 1/2$, $X_a = 3/4$, $X_a = 1$.*

Exercise 7.2.4. *: Show $Cov(r^a, r^b) = Cov(r^b, r^a)$.*

7.2.4 The general case

We will state but not prove the results for the general case. Suppose that the assets' rates of return take N possible values. Denote the assets by $a = 1, 2, \ldots, A$. Also, let X_a the share (percentage) of asset a in the portfolio. Then obtain:

$$Er^p = \sum_a X_a Er^a$$

$$Var(r^p) = \sum_a X_a^2 Var(r^a) + \sum_{a=1}^{A} \sum_{a' \neq a} X_a X_{a'} Cov(r^a, r^{a'})$$

7.2.5 The correlation coefficient in the two assets case

Suppose that the assets traded are a and b with rates of return defined above.

Definition 7.2.4. *: We define* ***the correlation coefficient*** *the following ratio:*

$$\rho_{a,b} = \frac{Cov(r^a, r^b)}{\sigma_a \sigma_b}.$$

Remark: if the rates of return of two assets are *independent* then their covariance is zero. Recall that two random variables r^a and r^b are said two be independent if $E(r^a r^b) = Er^a Er^b$. Then:

$$
\begin{aligned}
Cov(r^a, r^b) &= E[(r^a - Er^a)(r^b - Er^b)] = \\
&= E[r^a r^b - r^a Er^b - r^b Er^a + Er^a Er^b] = \\
&= E(r^a r^b) - 2Er^a Er^b + Er^a Er^b = \\
&= Er^a Er^b - 2Er^a Er^b + Er^a Er^b = 0.
\end{aligned}
$$

It follows that if r^a and r^b are independent then $\rho_{a,b} = 0$.
Remark: if the rates of return of two assets are *perfectly correlated* then the correlation coefficient equals 1: suppose that $r^a = C + Dr^b$ where C and D are two constants, then:

$$\begin{aligned} Cov(r^a, r^b) &= E[(r^a - Er^a)(r^b - Er^b)] = \\ &= E[(C + Dr^b - E(C + Dr^b))(r^b - Er^b)] = \\ &= E[(Dr^b - DEr^b)(r^b - Er^b)] = \\ &= DE[(r^b - Er^b)(r^b - Er^b)] = \\ &= DE(r^b - Er^b)^2 = D\sigma_b^2. \end{aligned}$$

Also: from property 3 of the variance we know that $\sigma_a^2 = D^2\sigma_b^2$ and then $\sigma_a = D\sigma_b$.

Then the correlation coefficient between the returns of asset a and asset b is given by: $\rho_{a,b} = \frac{Cov(r^a,r^b)}{\sigma_a\sigma_b} = \frac{D\sigma_b^2}{D\sigma_b^2} = 1$.

Exercise 7.2.5. *Show that if* $r^a = C - Dr^b$ *then* $\rho_{a,b} = -1$

Using the definition of correlation coefficient, we can rewrite the variance of a portfolio containing a portion X_a of asset a and X_b of asset b as follows:

$$\sigma_p^2 = X_a^2\sigma_a^2 + X_b^2\sigma_b^2 + 2X_aX_b\rho_{a,b}\sigma_a\sigma_b.$$

Remark: if $Cov(r^a, r^b) = 0$, then $\rho_{a,b} = 0$ and $\sigma_p^2 = X_a^2\sigma^a + X_b^2\sigma^b$.

Exercise 7.2.6. *: Suppose that the returns of two assets* a *and* b *have a correlation coefficient given by* $\rho_{a,b} = 1$*. Show that* $\sigma_p^2 = (X_a\sigma_a + X_b\sigma_b)^2$.

7.3 Mean-variance analysis and expected utility approach

In the mean-variance approach agents are assumed to care only about mean and variance of their consumption streams. The implications of such behavior on the optimal portfolio selection are quite important. In this section we want to show that the mean-variance analysis is equivalent to the expected utility approach (the approach taken up top now) under two sets of assumptions: 1) assumptions on the distribution of the rates of return; 2) assumptions on the agents' preferences.

7.3.1 Assumptions on the distribution of the rates of return

Consider the Von Neumann-Morgerstern representation of utility under uncertainty,

$$U(W) = \sum_{s=1}^{S} \pi_s u(W_s),$$

where W_s represents the individual income (or wealth) in state s. Given the set of probabilities $\{\pi_1, \ldots, \pi_S\}$, $U(W)$ is a function of the different outcomes, $\{W_1, \ldots, W_S\}$.

However, if we fix the set of contingencies $\{W_1, \ldots, W_S\}$ and the function $u(W)$, one can treat the expected utility function U as a function over the probability distribution. In this perspective, the expected utility representation is a special case of utility function over probabilities:

$$U(\pi_1, \ldots, \pi_S) = \sum_{s=1}^{S} \pi_s u(W_s)$$

Furthermore, since probability distributions are characterized by their moments (mean, variance, etc...) we can express the preferences of agents over the distribution's moments instead of the distribution itself.

Suppose in fact that $M = (M_1, M_2, \ldots, \ldots)$ represents the sets of moments of a distribution Π and that the state space is the entire real line. We can write the expected utility as follows:

$$V(M) = U(\pi_1, \ldots, \pi_S) = \int u(W) d\Pi(W|M).$$

As a consequence we find that if the distribution is entirely characterized by the first two moments (mean and variance), the preferences of the agents will only depend on the first two moments. Unfortunately, the normal distribution is the only (stable) distribution fully parameterized by the mean and the variance. Hence, assuming that agents care only about the first two moments of their portfolio is equivalent to assuming that the portfolio returns are normally distributed. Recall that the normal distribution has density function:

$$f(W; \mu, \sigma) = \frac{1}{\sqrt{2\pi\sigma}} \quad e^{-(W-\mu)^2/2\sigma^2}.$$

7.3.2 Assumptions on individual preferences

The alternative way to justify the assumptions underlining the mean-variance approach, is to restrict agents utility function to be quadratic (see Lecture 2), *i.e.:*

$$W - \frac{1}{2}W^2$$

In fact, recalling that $\sigma_W^2 = E(W^2) - (EW)^2$, we obtain that the quadratic utility function can be expressed as follows:

$$Eu(W) = EW - \frac{1}{2}E(W^2) =$$
$$EW - \frac{1}{2}(\sigma_W^2 + (EW)^2).$$

It is easy to see that the value of the function depends uniquely on the values of EW and σ_W^2. This satisfies the assumption that agents choose they portfolio composition only looking at the mean and variance of the portfolio returns.

Concluding, the mean-variance approach to portfolio selection is the state preferences approach with one extra condition, either on returns distributions or on individual preferences.

Lecture 8

THE MEAN-VARIANCE APPROACH

8.1 Introduction

The objective of this lecture is the description of traders' opportunity set in a mean-variance world, namely, the *portfolio frontier*, *i.e.,* the set of portfolios that a trader can select, given the available assets in the market. We will then identify the subset of portfolios that are efficient, namely, the *efficient frontier*. Next lecture we will introduce preferences and analyze the optimal portfolio choice.

8.2 The opportunity set in a mean-variance world

Given the assets available in the economy and the correlation of their returns, we will try to describe how the portfolio's expected value and standard deviation change when we change the portfolio's composition. If we are able to give a full description of these two variables, then we can draw some conclusion on the optimal portfolio choice.

Following the notation adopted in the last lecture, we consider a portfolio, say portfolio p, composed by a share X_a of asset a and a share $X_b = 1 - X_a$ of asset b.

We start by assuming a given correlation between the two assets. We then plot the combination of the values of the expected returns and values of the standard deviation that is attainable by changing each asset holding, *i.e.*: we will generate different portfolios by changing the values of X_a and see which risk-return combination can be attained. For this part of the analysis, let us rule out the possibility to shortsell, *i.e.,* having $X_a < 0$ and $X_b > 1$, or viceversa,

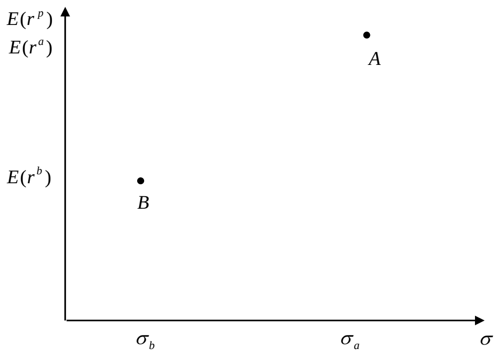
From Lecture 7, recall that:

$$\sigma_p^2 = X_a^2\sigma_a^2 + X_b^2\sigma_b^2 + 2X_aX_b\rho_{ab}\sigma_a\sigma_b$$

and

$$\begin{aligned} Er^p &= X_aEr^a + X_bEr^b \\ &= X_aEr^a + (1 - X_a)Er^b \\ &= Er^b + X_a(Er^a - Er^b). \end{aligned} \tag{8.1}$$

Notice that, independently of the value of $\rho_{a,b}$, if $X_a = 0$, then $\sigma_p = \sigma_b$ and $Er^p = Er^b$ and if $X_a = 1$, then $\sigma_p = \sigma_a$ and $Er^p = Er^a$ then the two points (Er^a, σ_a) and (Er^b, σ_b), must belong to the (Er^p, σ_p) frontier, irrespective to the value of ρ_{ab}. Call these points A and B, respectively. Graphically:



We proceed now by analyzing 2 cases when ρ_{ab} takes values $+1$ and -1.

8.2.1 Case 1) $\rho_{ab} = 1$.

Replacing the value of the correlation coefficient in the portfolio variance equation, obtain:

$$\sigma_p^2 = X_a^2\sigma_a^2 + X_b^2\sigma_b^2 + 2X_aX_b\sigma_a\sigma_b,$$

or:

$$\sigma_p = X_a\sigma_a + X_b\sigma_b = X_a\sigma_a + (1 - X_a)\sigma_b,$$

$$= \sigma_b + (\sigma_a - \sigma_b)X_a.$$

Solving for X_a obtain:

$$X_a = \frac{\sigma_p - \sigma_b}{\sigma_a - \sigma_b}.$$

Substitute back equation into (8.1) obtain:

$$Er^p = Er^b + \frac{\sigma_p - \sigma_b}{\sigma_a - \sigma_b}\left[Er^a - Er^b\right],$$

or:

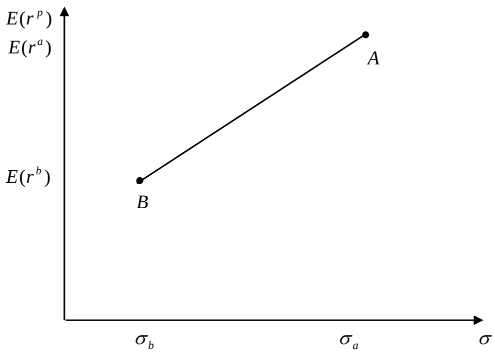
$$Er^p = \left[Er^b - \frac{\sigma_b}{\sigma_a - \sigma_b}(Er^a - Er^b)\right] + \frac{Er^a - Er^b}{\sigma_a - \sigma_b}\sigma_p.$$

This describes a linear relationship between Er^p and σ_p. Notice that there exists a zero variance portfolio but this is attainable only by shortselling either asset, something we have ruled out for this example. This follows from the fact that $\sigma_p = 0$ if and only if

$$X_a = \frac{\sigma_b}{\sigma_b - \sigma_a}.$$

This will lead either to $X_a > 1$ for $\sigma_b > \sigma_a$ and hence $X_b < 0$ or $X_a < 0$ for $\sigma_b < \sigma_a$ and hence $X_b > 1$.

We already had 2 points of the map: now we know that the locus must be described by the straight line joining these points, *i.e.*:



8.2.2 Case 2): $\rho_{a,b} = -1$

Recall that this case arises when the returns of the two assets are related by the formula of the type $r^a = C - Dr^b$.

We know that $\rho_{a,b} = -1$ implies:

$$\sigma_p^2 = X_a^2\sigma_a^2 + X_b^2\sigma_b^2 - 2X_aX_b\sigma_a\sigma_b.$$

There are two solutions to the equation for σ_p, namely:

$$\begin{aligned}\sigma_p &= \pm(X_a\sigma_a - X_b\sigma_b) = \pm(X_a\sigma_a - (1 - X_a)\sigma_b), \\ &= \pm(X_a(\sigma_a + \sigma_b) - \sigma_b) > 0. \end{aligned} \tag{8.2}$$

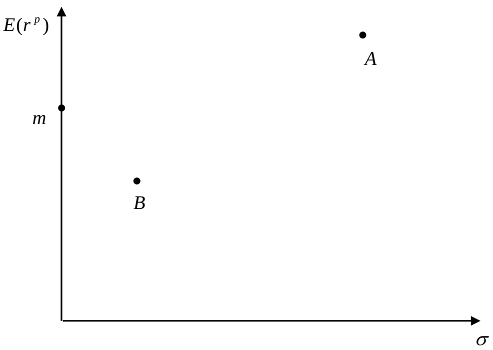
Notice that for given values of σ_a and σ_b, as X_a varies either one or the other solution will be positive but not both. Moreover:

$$\sigma_p(\rho_{ab} = -1) \leq \sigma_p(\rho_{ab} = 1)$$

Equation (8.2) implies that at:

$$X_a = \frac{\sigma_b}{\sigma_a + \sigma_b},$$

$\sigma_p = 0$, *i.e.,* when $\rho_{ab} = -1$ then there exists a riskless portfolio.
Let m be the point at $\sigma_p = 0$. Now, we have three points of the frontier, say (A, m, B), where m is the point of zero variance when $\rho_{ab} = -1$, graphically:



In order to describe the entire frontier for $\rho = -1$, we proceed as follows:

1) From the first (+) solution of σ_p in 8.2.2:

$$\sigma_p = X_a(\sigma_a + \sigma_b) - \sigma_b$$

obtain:

$$X_a = \frac{\sigma_b + \sigma_p}{\sigma_a + \sigma_b}$$

2) Do the same for the second solution (-) in 8.2.2 and obtain:

$$\sigma_p = -X_a\sigma_a + (1 - X_a)\sigma_b$$

hence:

$$X_a = \frac{\sigma_b - \sigma_p}{\sigma_a + \sigma_b}$$

3) Consider now the equation of the portfolio's expected return:

$$Er^p = X_a Er^a + (1 - X_a)Er^b = Er^b + X_a(Er^a - Er^b).$$

Substituting back the equation for X_a for the first solution obtain:

$$Er^p = Er^b + \frac{\sigma_b + \sigma_p}{\sigma_a + \sigma_b}\left[Er^a - Er^b\right]$$

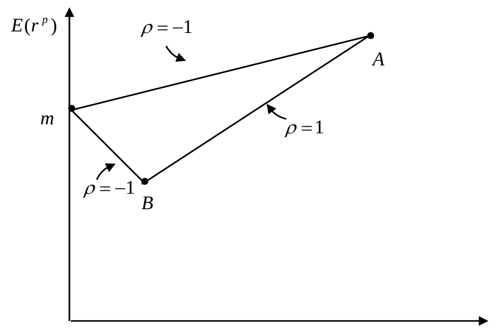
and rearranging it to obtain:

$$Er^p = Er^b + \frac{Er^a - Er^b}{\sigma_a + \sigma_b}\sigma_b + \frac{Er^a - Er^b}{\sigma_a + \sigma_b}\sigma_p.$$

For the second solution, obtain:

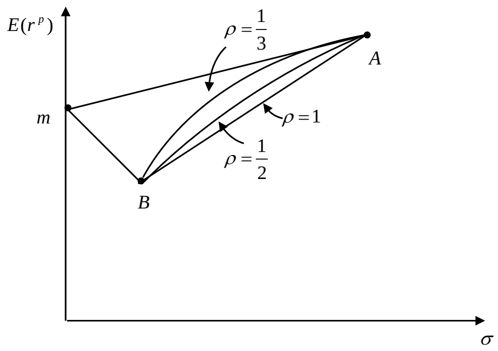
$$Er^p = Er^b + \frac{Er^a - Er^b}{\sigma_a + \sigma_b}\sigma_b - \frac{Er^a - Er^b}{\sigma_a + \sigma_b}\sigma_p.$$

Notice that both equations of the expected value are linear in the standard deviation. This implies that the frontier of portfolio mean-variance when $\rho_{ab} = -1$ is described by a linear equation. Plotting the cases analyzed until now we obtain:



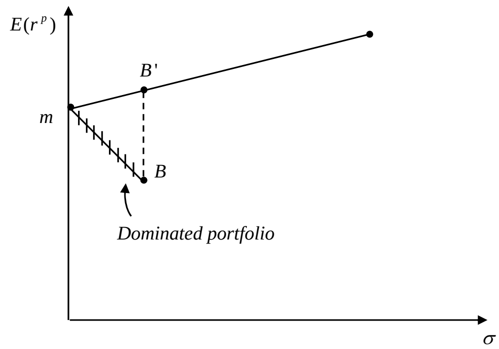
What happens in the case of general values of the correlation coefficient? It is possible to show that -irrespective to the value of ρ_{ab}- the locus of points (Er^p, σ^p) must be contained in the triangle AmB.

For different values of the correlation coefficient one would obtain the following plot:



8.3 The efficient frontier

Consider the next plot of the (Er^p, σ_p) equation, relative to a portfolio composed by assets with correlation $\rho_{ab} = -1$. The plot traces the set of available opportunities (*i.e.* the risk-return profile), ranging from the point A - representing (Er^p, σ_p) at total investments in asset a - to point B - representing (Er^p, σ_p) with total investments in asset b. Point m describes the portfolio with minimum variance.
For the risk averse investor, portfolios giving combinations (Er^p, σ_p) in the range mB are strictly dominated and hence inefficient. In fact, remark that for each portfolio in the mB range, there exits a portfolio on the mA range (say mB') that promises higher expected returns with a lower standard deviation.



8.4 Computing the minimum variance portfolio

From the definition of variance of a portfolio composed by asset a and b we obtain:

$$\sigma_p^2 = X_a^2\sigma_a^2 + (1 - X_a)^2\sigma_b^2 + 2X_a(1 - X_a)\rho_{ab}\sigma_a\sigma_b.$$

The value of σ_p is minimized where the first order condition for a minimum with respect to X_a is satisfied, *i.e.*,

$$\frac{\partial \sigma_p^2}{\partial X_a} = 2X_a\sigma_a^2 - 2(1 - X_a)\sigma_b^2 + 2\rho_{ab}\sigma_a\sigma_b - 4X_a\rho_{ab}\sigma_a\sigma_b = 0.$$

Then:

$$2X_a\sigma_a^2 - 2\sigma_b^2 + 2X_a\sigma_b^2 + 2\rho_{ab}\sigma_a\sigma_b - 4X_a\rho_{ab}\sigma_a\sigma_b = 0$$

$$X_a(2\sigma_a^2 + 2\sigma_b^2 - 4\rho_{ab}\sigma_a\sigma_b) - 2\sigma_b^2 + 2\rho_{ab}\sigma_a\sigma_b = 0$$

$$X_a = \frac{2\sigma_b^2 - 2\rho_{ab}\sigma_a\sigma_b}{2(\sigma_a^2 + \sigma_b^2 - 2\rho_{ab}\sigma_a\sigma_b)}.$$

In order to show that this is actually a minimum we need to verify that the second order condition of the problem (*i.e.,* the second derivative must be positive), *i.e.:*

$$\sigma_a^2 + \sigma_b^2 - 2\rho_{ab}\sigma_a\sigma_b > 0.$$

But this is always satisfied since the function $\sigma_a^2 + \sigma_b^2 - 2\rho_{ab}\sigma_a\sigma_b$ has a minimum at $\rho_{ab} = 1$ (*why*?). At the minimum value $(\sigma_a - \sigma_b)^2$ or $(\sigma_b - \sigma_a)^2$, two positive values.

Lecture 9

MEAN-VARIANCE CHOICE

"In plain English, an average investor -whether an individual, a pension fund, or a mutual fund- cannot hope to consistently beat the market, and the vast resources that such investor dedicates to analyzing, picking, and trading securities are wasted. Better to passively hold the market portfolio, and to forget active money management all together."

Andrei Shleifer[1]

"I am a principal in a money management firm. We do try to exploit these ideas [behavioral finance] to make money. We manage to beat our benchmark on average but not every year. It's not easy to make money! If you can outperform the market by 2 or 3% you are doing extremely well."

Richard Thaler[2]

"My advice to the trustee couldn't be more simple: Put 10% of the cash in short-term government bonds and 90% in a very low-cost S&P 500 index fund [...] I believe the trust's long-term results from this policy will be superior to those attained by most investors - whether pension funds, institutions or individuals - who employ high-fee managers."

Warren Buffett[3]

9.1 Introduction

In the last lecture we constructed the portfolio frontier in a two-asset world. Also, we showed that the lower portion of the portfolio frontier is dominated by the higher portion. In this lecture we generalize the analysis to the case with A risky assets and one risk-free security.

Subsequently, we introduce agents' preferences. Recall the analysis of consumption choice that we developed in the first part of the course. Preferences of agents were defined over different bundles of commodities. In this lecture we assume that agents' preferences are defined over the mean and variance of assets returns as a proxy for consumption and risk. We assume that the actual underlining preferences of agents are monotonically increasing in the consumption good and that they are convex, *i.e.,* agents are risk averse as the utility function is strictly concave .

The objectives of this lecture are the following:

[1] "Inefficient Markets: An introduction to behavioral finance," Claredon Lectures in Economics, OUP, 2000.
[2] Interview with the Financial Times, April 29th, 2009.
[3] Annual letter to Berkshire shareholders, Feb. 28 2014.

1. The derivation of the efficient frontier in a general set up.

2. The characterization the optimal portfolio choice given the agent's preferences: the argument that we follow is basically the same used in the consumption choice problem. The rational investor maximizes his utility function subject to the available combinations of risk and variance.

3. The two-fund separation theorem: along with the CAPM (see later) this is the fundamental result of the second part of the course.

4. The analysis of the variance of the market portfolio as the number of assets increases.

9.2 The efficient frontier in the general case

Let us start by deriving the efficient frontier when there are only two assets, one risky and the other riskless. We will then generalize the analysis to the case of one rikless and multiple risky assets.

9.2.1 One risk-free and one risky asset

Consider a two-asset economy. Let the first asset be asset a and suppose that the second asset, call it asset f, is risk-free: $\sigma_f = 0$. Also, assume that short selling of both assets are allowed: the investor can borrow (*i.e.,* short-sell asset f) and hold more than his wealth in asset a or she can short-sell asset a and buy the risk free asset f. In this case, the portfolio variance reduces to$\sigma_p^2 = X_a^2 \sigma_a^2$ with standard deviation given by $\sigma_p = |X_a|\sigma_a$. Solving for X_a obtain:

$$|X_a| = \frac{\sigma_p}{\sigma_a}$$

or:

$$X_a = \pm \frac{\sigma_p}{\sigma_a}$$

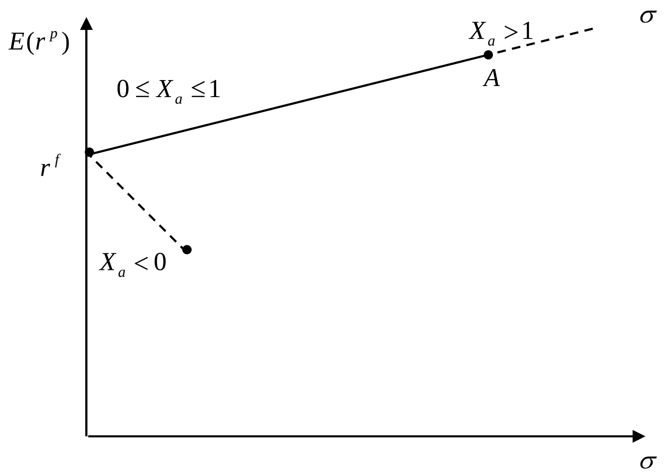
Substitute now the two possible values of X_a in the expected rate of return of the portfolio, equation (8.1). For the positive value obtain:

$$Er^p = \frac{\sigma_p}{\sigma_a} Er^a + (1 - \frac{\sigma_p}{\sigma_a}) r^f \quad \Rightarrow \quad Er^p = r^f + \frac{Er^a - r^f}{\sigma_a} \sigma_p,$$

and for the negative value obtain:

$$Er^p = -\frac{\sigma_p}{\sigma_a} Er^a + (1 + \frac{\sigma_p}{\sigma_a}) r^f \quad \Rightarrow \quad Er^p = r^f - \frac{Er^a - r^f}{\sigma_a} \sigma_p.$$

This gives a linear relationship in the $(E(r^p), \sigma_p)$ space. Plotting the above functions obtain:

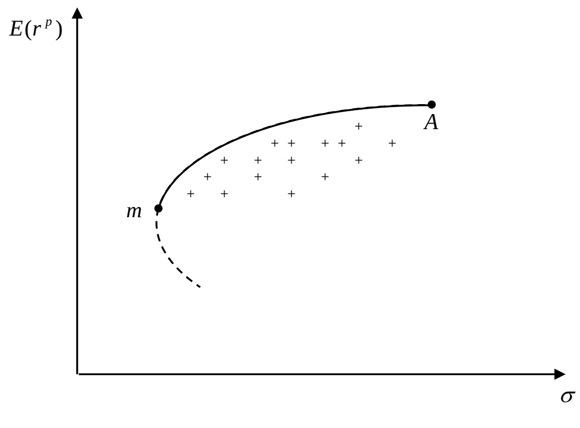


Notice that the portfolios along the segment $r^f - A$ the investor is allocating his wealth between the two assets *i.e.,* he is holding a positive amount of the two assets. The negatively slope line beyond r^f, the investor is holding more than 100% of his wealth of the risk-free asset and short-selling the risky asset $X_a < 0$. Vice-versa for the portion of the frontier beyond point A.

9.2.2 The efficient frontier with A risky assets

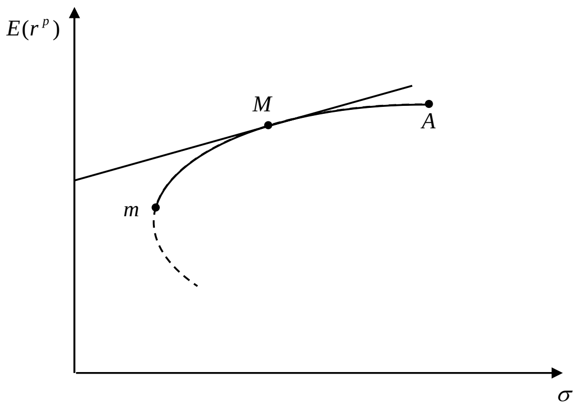
We claim, but do not prove, that in the case of several assets the portfolio frontier has the same shape of the frontier in the case of a two assets world. In this case, there will be several assets that will be strictly dominated by some portfolio, nevertheless, typically, some share of them will be held by the investor.

The points below the efficient frontier denote the assets/portfolios that are dominated, *i.e.,* there is another feasible portfolio that promises the same return at a lower variance.



9.2.3 One risk-free asset and A risky assets

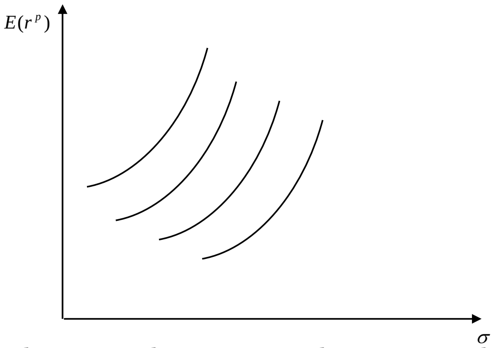
Suppose now along with the many risky assets, we introduce one risk-free asset (one is enough, *why?*).
Let us describe the portfolio frontier for different combinations of risky portfolios with the riskless asset. From the figure below, it is easy to see that once the risk-free asset has been introduced, only the portfolio M on the frontier mA will be selected: all other combinations of the asset f and the risky portfolio are strictly dominated by some point on the line from $(0, r^f)$ to (σ_M, Er^M).



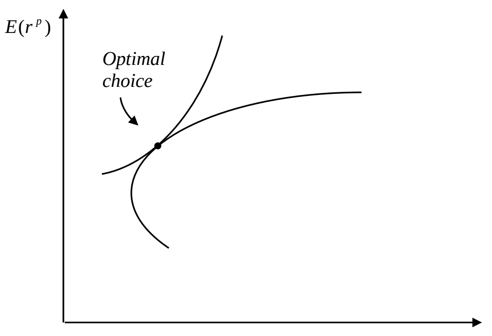
The tangent at M is called the ***Capital Market Line*** (CML) and represents the is the set of undominated portfolios The slope of the CML is given by $\frac{Er^M - r^f}{\sigma_M}$ (why?).

9.3 The optimal portfolio choice

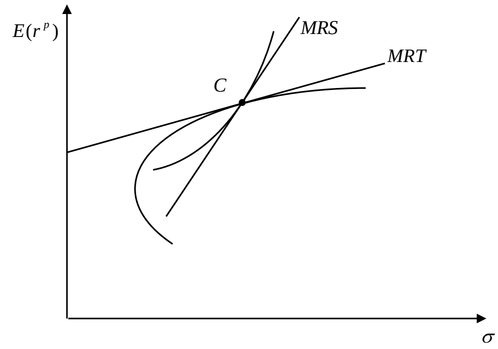
Until now, we haven't mentioned investors preferences. In order to complete the analysis of optimal portfolio selection in a mean-variance world, we need to see how preferences affect investors' choices.
Consider an agent with preferences over (Er^p, σ_p) and assume that the agent is risk averse and prefers higher to lower returns. This implies that the agent indifference curves are upward sloping: when σ_p increases, $E(r^p)$ must also increase in order to stay on the same indifference curve. Moreover, we will assume that the agent risk aversion increases with the level of risk: the higher σ_p the higher the increment of return required in order to stay on the same indifference curve (*which is the equivalent concept is standard micro theory?*).
Notice that the utility levels increase from east to west (by risk aversion) and from south to north (by monotonicity).



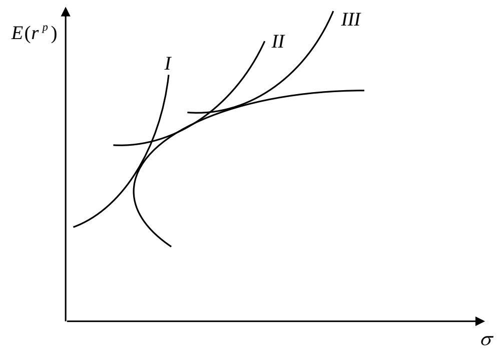
The optimal point is at the tangency between between the indifferent curve and portfolio frontier.



Suppose not. Consider then an investor holding a portfolio given by point C in the next figure. Looking at the MRS at the portfolio C, we see that the agent is willing to give up a relatively high portion of return in order to reduce his risk by a relatively small amount (the slope of his indifference curve at C is quite high). On the contrary, the market allows him to give up a lower amount of return for the same reduction of risk: $MRS_C > MRT_C$. So the investor will be better off by trading returns for risk by moving along the portfolio frontier and C cannot be an optimal choice. He will keep moving in the same direction up to the point where $MRS_C = MRT_C$.



Remark that different investors will select different portfolios: a less risk averse individual will end up choosing a portfolio with higher risk than a more risk averse investor, this in exchange for a higher expected return. The following graph plots the choice of three different investors, namely I, II and III, with different levels of risk aversion.

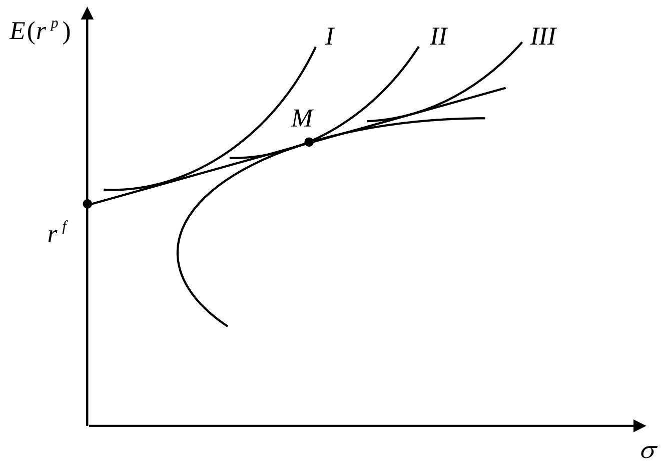


9.4 The Two-Fund Separation theorem

Assumption 9.1. *1.* ***(mean-variance world)*** *Investors rank portfolios only on the basis of their mean and variance.*
2. ***(perfect capital markets)*** *The borrowing and lending rates are the same.*
3. ***(no asymmetric information)*** *Investors hold homogeneous beliefs about returns of the asset traded.*

Exercise 9.4.1. *Under Assumption 9.1.2 and 9.1.3 the Capital Market Line is the same across investors. Why?*

Consider the optimal choice of three investors with different preferences.



Investor I, the most risk-averse, holds a share of his wealth in portfolio M and the remaining share in asset f, *i.e.,* lends some amount of his wealth. Investor II, holds only the risky portfolio M, neither borrows nor lends. Investor III borrows the risk-free asset in order to invest more than his wealth in the risky portfolio. Since the CML (joining r^f with M) is the same for all investors, what distinguishes the three investors is only the different shares of portfolio M and asset f, *i.e.,* no other risky portfolio is held but M. This fact is known as the two-fund separation theorem.

Theorem 1. ***The two-fund separation theorem*** *Given the assumptions 1-3, the optimal portfolio consists of a mix of the riskless asset and the market portfolio.*

According to the two-fund separation theorem, hence, equilibrium allocations would not change if we replace the market portfolio (a composition of assets) with a unique asset promising the same expected return with the same variance.

The two-fund separation theorem is a reinterpretation of the Fisher separation theorem in a world with mean and variance uncertainty. It suggests the same conclusions on separation of ownership and control.

9.5 Portfolio diversification and individual asset risk

In this section we want to show that as the number of assets increases (*i.e,* A increases), the portfolio variance approaches the *average covariance.*
Recall that the formula of the variance of a portfolio composed by A assets is given by:

$$\sigma_p^2 = \sum_{a=1}^{A} X_a^2 \sigma_a^2 + \sum_{a=1}^{A} \sum_{a' \neq a} X_a X_{a'} Cov(r^a, r^{a'}).$$

Suppose $X_a = X_{a'} = A^{-1}$ for all assets a and a' (equal percentage is invested in each asset). The above formula becomes:

$$\sigma_p^2 = A^{-2} \sum_{a=1}^{A} \sigma_a^2 + A^{-2} \sum_{a=1}^{A} \sum_{a' \neq a} Cov(r^a, r^{a'}).$$

Now let:

$$L = \max\{\sigma_a^2 : a = 1, \ldots, A\},$$

i.e., L is the highest variance across all assets.[4]
Then:

$$A^{-2}\sum_a \sigma_a^2 \leq A^{-2}(AL) = \frac{L}{A}.$$

This implies that:

$$\lim_{A\to\infty} \frac{L}{A} \to 0,$$

and hence:

$$A^{-2}\sum_{a=1}^{A} \sigma_a^2 \to 0$$

i.e: the weighted sum of variances tends to zero as the number of assets increases.

However, this is not true for the covariance terms. In fact, let $\overline{Cov}$ denote the average covariance. The total number of covariance terms are given by $(A^2 - A)$: A^2 is the number of possible products across assets (the total number of elements in the variance-covariance matrix) minus the covariance of each asset with itself, *i.e.,* the elements on the diagonal in the variance-covariance matrix.
It follows that the part the variance equation involving covariance terms can be written as:

$$\frac{A^2 - A}{A^2}\overline{Cov} = (1 - \frac{1}{A})\overline{Cov}$$

Hence, for A increasing obtain:

$$\lim_{A\to\infty}(1 - \frac{1}{A})\overline{Cov} = \overline{Cov}.$$

i.e., the sum of the weighted covariances tends to the average covariance as the number of assets increases. This implies that:

$$\lim_{A\to\infty} \sigma_p^2 = \overline{Cov}.$$

Exercise 9.5.1. *What is the effect on agents' portfolio variance if traders are allowed to invest in foreign assets?*

[4] L must exist as the returns of all assets are normally distributed.

Lecture 10

THE CAPITAL ASSET PRICING MODEL

10.1 Introduction

In the first part of the course, we learned how to price assets at equilibrium using a general equilibrium model in which agent preferences where defined over consumption bundles and uncertainty was represented by a set of possible states of nature. The theory needed very weak assumptions on preferences, beliefs, probability distributions over the state space, etc. Furthermore, this theory has robust extensions to imperfect markets, asymmetric information, financial innovation, moral hazard, bankruptcy, real effects of money, etc.

In the second part of the course we introduced mean-variance models and we studied agents' portfolio selection. In this lecture we will complete the model by showing how to determine the rates of return at equilibrium.

10.2 CAPM

Consider an economy satisfying the following assumptions:

Assumption 10.1. *1) The agents are risk averse and maximize their end of the period wealth;*
2) The asset market is perfectly competitive: each investor takes prices as given;
3) The agents' beliefs are homogeneous, i.e.: there is no asymmetric information or equivalently, information is cost-less;
4) The asset returns have a joint normal distribution;
5) There is no financial innovation;
6) The assets are perfectly divisible and marketable;
7) The asset market is friction-less.

Assumptions 2, 6 and 7 are in order to guarantee the existence of a market equilibrium. If the market for each asset clears all asset will be held at equilibrium as asset prices move in order to make demand and supply equal. At equilibrium, the proportion of each asset a in the market portfolio is given by the ratio of the value of asset a and the market value of the portfolio.
So, how are assets' expected returns determined in a mean-variance model? We start by considering the equilibrium market portfolio M with expected return Er^M and variance σ_M^2. Suppose we introduce a new portfolio, say portfolio p, composed of X_a percent of an asset, say asset a, and $(1 - X_a)$ percent of portfolio M. So the portfolio p is a combination of asset a with portfolio M. We start by computing how the expected return and the variance of the new portfolio change as X_a increases. Computing Er^p and σ_p obtain:

$$Er^p = X_a Er^a + (1 - X_a) Er^M$$

$$\sigma_p = [X_a^2\sigma_a^2 + (1-X_a)^2\sigma_M^2 + 2X_a(1-X_a)Cov(r^a, r^M)]^{1/2}.$$

Taking the derivative of the above values with respect to X_a we obtain:

$$\frac{\partial Er^p}{\partial X_a} = Er^a - Er^M$$

$$\frac{\partial \sigma_p}{\partial X_a} = \frac{1}{2}[2X_a\sigma_a^2 - 2(1-X_a)\sigma_M^2 + 2(1-X_a)Cov(r^a, r^M) - 2X_aCov(r^a, r^M)].\times$$

$$\times[X_a^2\sigma_a^2 + (1-X_a)^2\sigma_M^2 + 2X_a(1-X_a)Cov(r^a, r^M)]^{-1/2}$$

Evaluating this equation at equilibrium, *i.e.*, where $X_a = 0$:

$$\frac{\partial Er^p}{\partial X_a}|_{X_a=0} = Er^a - Er^M;$$

and

$$\frac{\partial \sigma_p}{\partial X_a}|_{X_a=0} = \frac{1}{2}\frac{[-2\sigma_M^2 + 2Cov(r^a, r^M)]}{(\sigma_M^2)^{1/2}} =$$

$$= \frac{1}{2}\frac{2(Cov(r^a, r^M) - \sigma_M^2)}{\sigma_M} =$$

$$= \frac{Cov(r^a, r^M) - \sigma_M^2}{\sigma_M}.$$

The slope of the risk-return trade-off evaluated at $X_a = 0$ is given by:

$$\frac{\partial Er^p/\partial X_a}{\partial \sigma_p/\partial X_a} = \frac{Er^a - Er^M}{Cov(r^a, r^M) - \sigma_M^2}\sigma_M.$$

At equilibrium this slope must be the same irrespective to the asset considered.
In fact suppose that for two assets, say asset a and b, this is not true but the slopes are such that:

$$\frac{\partial Er^p/\partial X_a}{\partial \sigma_p/\partial X_a} > \frac{\partial Er^p/\partial X_b}{\partial \sigma_p/\partial X_b}$$

then there exists a reallocation of assets delivering a portfolio that gives a higher return for the same risk.

Example 3. *: Suppose at equilibrium the following equalities are verified:*

$$\frac{\frac{\partial Er^p}{\partial X_a}}{\frac{\partial \sigma_p}{\partial X_a}} = \frac{10}{2} \text{ and } \frac{\frac{\partial Er^p}{\partial X_b}}{\frac{\partial \sigma_p}{\partial X_b}} = \frac{8}{2}.$$

Notice that a decrease in X_b decreases the variance of the portfolio by 2 and its expected return by 8. However, an increase of X_a increases the variance by 2 and the expected return of the portfolio by 10. A portfolio containing a marginal increase of asset a and a marginal decrease of asset b would have the same variance of the original portfolio but higher returns. This is a contradiction as the original portfolio would be an equilibrium portfolio. Hence, all mean and variance trade-offs must be equal at equilibrium.

Furthermore, the ratios above must equal the slope of the capital market line as those ratios represent the tangency to the efficiency frontier exactly. Recall that the slope of the CML is given by:

$$\frac{Er^M - r^f}{\sigma_M}.$$

It follows that equilibrium, the following equality must hold:

$$\frac{Er^M - r^f}{\sigma_M} = \frac{Er^a - Er^M}{Cov(r^a, r^M) - \sigma_M^2}\sigma_M$$

$$\Rightarrow Er^a = \frac{Er^M - r^f}{\sigma_M^2}(Cov(r^a, r^M) - \sigma_M^2) + Er^M$$

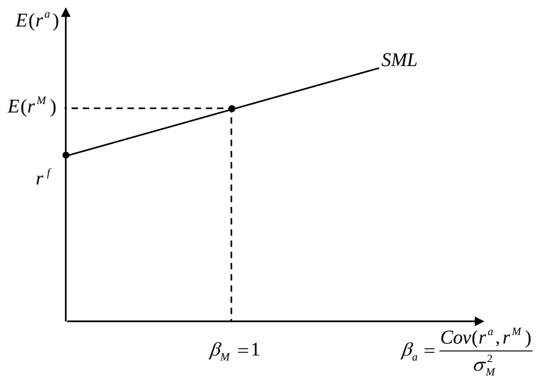
$$= \frac{Er^M - r^f}{\sigma_M^2}Cov(r^a, r^M) - Er^M + r^f + Er^M,$$

and finally:

$$\boxed{Er^a = r^f + [Er^M - r^f]\frac{Cov(r^a, r^M)}{\sigma_M^2}.}$$

The last equation describes the equilibrium return of each asset a according to the CAPM. The only component that distinguishes the expected returns across assets is the asset *covariance* with the portfolio and the variance plays no role in determining the equilirbrium return! The ratio $\frac{Cov(r^a, r^M)}{\sigma_M^2}$ is called the beta of asset a, denoted by β_a.

The figure below plots Er^a with β_a.



The straight line is called the security market line (SML). Also, notice that $\beta_M = 1$ and $\beta_f = 0$ (where f denotes the risk-free asset).
A convenient property is that the beta of a portfolio of assets is simply given by the weighted average of the betas of the assets it contains. Suppose that at equilibrium, we decide to compose a new portfolio of two existing assets, say a and b, with respective shares and betas given by X_a, β_a and X_b, β_b. Then:

$$\beta_p = X_a\beta_a + X_b\beta_b.$$

Exercise 10.2.1. *Prove the above linearity of β.*

10.3 Valuation by CAPM: two-periods models.

Assume the economy lasts two periods only. By CAPM we can value an asset that has a risky payoff at the second period.[1]
Let q_a be the price of the asset and q_a^e its (expected) price in the second period. The expected rate of return of the asset is defined by:

$$Er^a = \frac{q_a^e - q_a}{q_a}$$

What is the equilibrium price for the asset? The CAPM formula tells us that:

$$Er^a = r^f + (Er^M - r^f)\frac{Cov(r^a, r^M)}{\sigma_M^2}$$

or:

$$Er^a = r^f + \lambda Cov(r^a, r^M)$$

where the value $\lambda = \frac{Er^M - r^f}{\sigma_M^2}$ represents the shadow price for risk.
By equating the computed expected returns obtain:

$$\frac{q_a^e - q_a}{q_a} = r^f + \lambda Cov(r^a, r^M)$$

and solving for q_a obtain the *risk-adjusted rate of return valuation formula* according to CAPM:

$$\boxed{q_a = \frac{q_a^e}{1 + r^f + \lambda Cov(r^a, r^M)}}$$

[1]Notice that some textbook, e.g. Copeland and Weston, call this type of model single period models, as the second period is interpreted as the end of the first period.

THE NOTATION IS THE SAME AS IN THE NOTES

PROBLEM SET 1

1. Consider the problem of an agent living in a one-good, two-period economy without uncertainty. Let her endowment be equal to e_0 and e_1 in the first and second period, respectively and denote her utility function by $U(x_0, x_1)$ where x_0 and x_1 is the consumption in the first and second period, respectively. Suppose the agent has available a production technology described by the function $f(z)$ where z denotes the input of the good. Finally, suppose that a bond with return r is available in the financial markets at price q.

   (a) Assume the production technology is such that $f(0) = 0$, $f' > 0$ and $f'' < 0$ with $f'(0) > \frac{r}{q}$ and the endowment is strictly positive, i.e. $e_0 > 0$ and $e_1 > 0$. Show that the optimal production choice is independent of the agent's preferences.

   (b) Suppose now $e_0 = 0$ (with $e_1 > 0$). In a graph identify the optimal consumption, production and investment choice.

   (c) Suppose now that the production technology is linear, i.e. $f'(z) = 1 - d$ for all values of z where d is constant and such that $(1 - d) < \frac{r}{q}$. What is the optimal level of production in this case? What is the value of the firm in this case? Can you give an example of such a technology?

2. Consider an agent living in a two-period, one-good, riskless economy with production and financial markets like the one analyzed in Lecture 1. Suppose the agent is endowed with a production technology represented by $2z - \frac{1}{2}z^2$ and she can borrow or lend in the financial market at a rate equal to $\frac{r}{q} = 1$ (q is the price of the borrowing/lending contract and r the gross return). The agent has preferences represented by the utility function $\ln x_0 + \ln x_1$. Her endowment at $t = 0$ is given by 10 units of the good and her endowment at $t = 1$ is denoted by e_1.

   (a) For which value of e_1 the agent is neither a lender nor a borrower?

   (b) What is the value of the firm?

3. Consider the following cases of different production technologies in an economy where the interest rate is denoted by $\frac{r}{q}$.

   (a) The technology, described by $f(\cdot)$ with $f'(\cdot) > 0$ and $f''(\cdot) < 0$, requires input z only at $t = 0$ (so nothing at $t = 1$) and delivers the output at $t = 2$. Which conditions are needed on the function f in relation to the interest rate for the technology to be viable? What is the value of the firm at $t = 0$?

   (b) Suppose now the economy lasts T periods, denoted by $t = 0, 1 \ldots, T$. The production is such that at each t an input z delivers output $f(z)$ at $t+1$. Which conditions need to hold on $f(\cdot)$ for the firm to be viable at any t and what is the present value of the firm?

PROBLEM SET 2

1. Consider three individuals with wealth W and utility functions given by :

$$u^1(W) = 4W^{\frac{1}{4}},$$

$$u^2(W) = aW - \frac{1}{2}W^2, \text{ with } W < a,$$

$$u^3(W) = -\frac{1}{\alpha}\exp(-\alpha W),$$

where $\exp(\cdot)$ denotes the exponential function.

Compute their respective coefficient of absolute risk aversion (ARA) and relative risk aversion (RRA) and show how they change with the changes in wealth (i.e., compute whether the respective ARA and RRA are decreasing, costant or increasing).

2. Consider an economic agent with utility function $u = \sqrt{W}$ where W denotes wealth. The agent's initial wealth is given by 36. The agent is facing a zero price lottery with two possible outcomes/states, 1 and 2, with value -11 and +13, respectively. The probability of outcome 1 and 2 is given by α and $1 - \alpha$, respectively.

   (a) For which value of α (denote it α^*) is the lottery a fair gamble?

   (b) If $u(W) = \sqrt{W}$, for which maximum value of α (denote it α^{**}) will the agent enter the lottery?

   (c) If $\alpha = \frac{1}{2}$, what is the agent's risk premium associated to this lottery?

3. Consider now the same individual of question 2 above and suppose now that she is actually owning the lottery ticket described in the question. Suppose that she is offered to buy another zero price lottery ticket and that would give her +5.25 and -6.75 in state 1 and 2, respectively. If $\alpha < \frac{1}{2}$, do you think the agent would be willing to pay to buy the ticket? If so, why? No points without explanation.

4. Consider a simple two-period economy with two possible states of the world in the second period. Financial markets are active and three assets are traded. The matrix of asset returns is given by (columns refer to assets, rows to states of nature):

$$\begin{bmatrix} 2 & 3 & 4 \\ 2 & 2 & 4 \end{bmatrix}.$$

Asset prices are 2, 4 and 5 for the first, second and third asset, respectively. State whether in such an economy there exists an arbitrage portfolio, whether the low of one price is violated and/or whether the no-arbitrage condition is violated. Explain (i.e., prove) you answer.

5. Consider an economy under uncertainty. There are two possible states of the world tomorrow and traders trade in two asset with the following returns:

$$\begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}.$$

The prices of the first and second asset are 1 and 2 respectively. Compute the risk neutral probabilities and discount factor.

6. Consider a simple two-period economy. In the second period there are two possible contingencies, states of nature, denoted by $s = 1, 2$. The financial markets are complete and the matrix of asset returns is given by (columns refer to assets, rows to states of nature):

$$\begin{bmatrix} 2 & \alpha \\ 3 & 1 \end{bmatrix}$$

(a) For which values of α are the markets complete?

(b) Suppose now that the price of the Arrow security 1 (the security that pays 1 in state 1 only) is given by 1. If the price of assets 1 and 2, are 8 and 4 respectively, what is the value of α?

(c) What is the value of the risk neutral probabilities in the economy of question 6b above?

PROBLEM SET 3

1. Consider an economy extending over two periods, $t = 0, 1$ under uncertainty. There are two agents in the economy, denoted by $h = 1, 2$ and two states of nature in the second period, denoted by $s = 1, 2$ to which agents assign probability π_s. There is only one good in each period and contingency. The wealth distribution is represented by the following matrix:

Commodity	Agent 1	Agent 2
t=0	10	10
State 1	e_1^1	10
State 2	e_2^1	20

   (a) Assign values to e_1^1 and e_2^1 such that:

      i. the econony does not experience aggregate risk but agents experience idiosyncratic risk;

      ii. there is no idiosyncratic risk but there is aggregate risk.

   (b) In which of the two cases identified in questions 1(a)i and 1(a)ii would risk averse agents find useful to trade a full set of Arrow securities?

2. Consider an economy extending over two periods, $t = 0, 1$ under uncertainty. There are two, equally probable states of nature in the second period, denoted by $s = 1, 2$. There are two individuals in the economy, denoted by $h = 1, 2$. Suppose there is one consumption good in the first period but two consumption goods in the second. The individuals have equal preferences given by: $x_0^h + E(\ln x_{s,1}^h + \ln x_{s,2}^h)$, where x_0^h denotes individual h's consumption in the first period and $x_{s,c}^h$ denotes h's consumption in state $s = 1, 2$ of commodity $c = 1, 2$.

   The first and second individual have 5 and 10 units of the good in the first period, respectively (*i.e.,* $e_0^1 = 5$ and $e_0^2 = 10$). In the first state, individual 1 has 10 units of the first good and 20 units of the second, while he has nothing in the second state. The endowment of the second individual is of 20 units of the first and second commodity in the second state, but nothing in the first state. Summarizing: the endowment of the first individual is given by $(e_0^1, e_{1,1}^1, e_{1,2}^1, e_{2,1}^1, e_{2,2}^1 = 5, 10, 20, 0, 0)$ and for the second individual this is given by $(e_0^2, e_{1,1}^2, e_{1,2}^2, e_{2,1}^2, e_{2,2}^2 = 10, 0, 0, 20, 20)$

   A full set of Arrow securities is traded in the financial market. Let y_a^h the demand of agent h for asset a. In order to simplify notation, normalize the price of the first good in each period and state, *i.e.,* let $p_0 = 1$ the price of consumption in the first period and $p_{1,1} = p_{2,1} = 1$ the prices of the consumption of the first good in each state.

   (a) How many goods are in this economy? Explain why the number of goods is higher than the number of commodities.

   (b) Explain why the chosen normalization of prices is without loss of generality. Is this the only normalization that we could have choosen?

   (c) Explain which fundamental assumption we are introducing in this dynamic economy and how it differs from the minimal assumptions needed to compute the "standard" equilibrium of a static economy (i.e., the Walrasian Equilibrium).

   (d) Write the budget constrain of each agent.

PROBLEM SET 4

1. Consider an economy extending over two periods, $t = 0, 1$ under uncertainty. There are two, equally probable states of nature in the second period, denoted by $s = 1, 2$. There are two individuals in the economy, denoted by $h = 1, 2$. Suppose there is one consumption good in the first period but two consumption goods in the second. The individuals have equal preferences given by: $x_0^h + E(\ln x_{s,1}^h + \ln x_{s,2}^h)$, where x_0^h denotes individual h's consumption in the first period and $x_{s,c}^h$ denotes h's consumption in state $s = 1, 2$ of commodity $c = 1, 2$.

   The first and second individual have 5 and 10 units of the good in the first period, respectively (*i.e.,* $e_0^1 = 5$ and $e_0^2 = 10$). In the first state, individual 1 has 10 units of the first good and 20 units of the second, while he has nothing in the second state. The endowment of the second individual is of 20 units of the first and second commodity in the second state, but nothing in the first state. Summarizing: the endowment of the first individual is given by $(e_0^1, e_{1,1}^1, e_{1,2}^1, e_{2,1}^1, e_{2,2}^1 = 5, 10, 20, 0, 0)$ and for the second individual this is given by $(e_0^2, e_{1,1}^2, e_{1,2}^2, e_{2,1}^2, e_{2,2}^2 = 10, 0, 0, 20, 20)$

   A full set of Arrow securities is traded in the financial market. Let y_a^h the demand of agent h for asset a. In order to simplify notation, normalize the price of the first good in each period and state, *i.e.,* let $p_0 = 1$ the price of consumption in the first period and $p_{1,1} = p_{2,1} = 1$ the prices of the consumption of the first good in each state.

   (a) Compute the equilibrium of this economy.
   (b) Show the equilibrium is Pareto efficient.

2. Consider an economy extending over two periods, under uncertainty. Uncertainty is represented by two possible states of the world in the second period, denoted $s = 1, 2$, occurring with equal probabilities. There are two agents and only one good is consumed at each period and in each contingency. Denote by x_c the consumption of commodity $c = 0, 1, 2$ where $c = 0$ denotes first period, $c = 1$ and $c = 2$ denote the second period's first and second state, respectively.

   Agents' preferences are identical and represented by the utility function:

   $$x_0 + x_1 - \frac{1}{2}(x_1)^2 + x_2 - \frac{1}{2}(x_2)^2.$$

   In the first period, each agent is endowed with 1 unit of the good.

   In the second period, the first agent's endowments are given by 0.4 and 0.3 units of the good in the first and second state, respectively. Similarly, the second agent's endowments in the second period are given by 0.2 and 0.1 units of the good in the first and second state, respectively. Suppose initially that an asset with a (strictly positive) vector of returns (r_1, r_2) is traded.

   (a) Compute the equilibrium of the economy.
   (b) Assume now that an option is introduced on the already traded asset and the strike price given by c_1, where $r_1 < c_1 < r_2$.
      i. Compute the equilibrium of the economy.
      ii. Is this equilibrium Pareto optimal. If so, why?
      iii. Would the equilibrium allocation change if instead the strike price was different, i.e., c_2 instead of c_1?

3. **Informationally efficient economies**

   Consider a two-period economy ($t = 0, 1$) under uncertainty. In the second period there are two states of nature, $s = H, L$ (H stands for high and L for low). Suppose one consumption good is traded at each period and at each state of the world.

   There are two individuals with equal preferences given by $\ln x_0 + E \ln x$, where x_0 denotes consumption in the first period and x denotes consumption (a random variable) with realization x_H and x_L in state H and L, respectively. Agents' endowments are as follows: in the first period each agent has 10 units of the good; in state H the first agent has an endowment of 40 units and the second agent has an endowment of 0 units; in state L, the first agent has an endowment of 0 units and the second agent an endowment of 20 units.

   Agents are asymmetrically informed about the future realization of the state of the economy. The first agent is better informed and knows the future state with certainty. Agent 2 (initially) has no information and assigns probability 0.5 to each state but knows that prices might reflect the information other agents in the market might have.

   Suppose that agents exchange one bond promising a return of 1 unit of consumption good in both states (*i.e.,* the vector of returns is (1,1)).

   (a) Find the *fully revealing equilibrium* of this economy.

   (b) Could the price of the asset be revealing if the returns were (2,1)? Explain.

   (c) Could the price of the asset be revealing if the informed agent's endowment in the second period were $e^1_H = e^1_L = 10$? Explain.

4. **The "Hirshleifer effect"** Consider a simple two-period, two-agent economy under uncertainty. Uncertainty is represented by two possible realizations/states of the world, denoted $s = 1, 2$, equally probable occurring at period 2. In the first period there is no consumption but agents trade two Arrow securities promising returns (1,0) and (0,1) respectively. Let the price of the first security be the numeraire and call q the price of the second security. In the second period agents consume the only available good. The first agent has 200 units of the good if the first state occurs but only 1 if the second state occurs. Viceversa, the second agent has only 1 unit of the good if the first state occurs but 200 units of the good in state 2. Agents' preferences for the consumption x of the good are represented by utility function $\ln x$ (i.e., the expected utility function is given by $\frac{1}{2} \ln x_1 + \frac{1}{2} \ln x_2$, where x_s denotes second period consumption in state $s = 1, 2$).

   (a) Compute the equilibrium for this economy.

   (b) Suppose now that a third agent proposes to announce publicly the state of the world *before* agents trade in the asset. Would the agents be willing to receive this information?

   (c) What can you conclude about the value of public information?

   (d) If prices are informational efficient, do you think that the answer to question (4b) above would change if the third agent proposes to give the information to only one of the two agents instead of announcing it publicly?

   (e) Suppose that the third agent proposes to announce the true state of the world in period 0 but *after* the trade in the asset has taken place. What is the value of information in this case?

(Hints: Recall that the ln is a concave function (*i.e.,* preferences are convex). Useful numbers: $\ln 1 = 0$, $\ln 200 \approx 5.3$, $\ln 100.5 \approx 4.6$)

PROBLEM SET 5

1. Consider a firm with an investment opportunity of value 100. The net present value of the investement depends on the quality of the firm, that can be of two types, H or L. If a firm is of quality H its value (assets-in-place) is 400 and the investment has a return of 40%. If the firm is of type L its value is 80 and the investment rate of return is 20%. Suppose that the only way the firm has to finance the project is to issue new equity on the financial market.

   The firm knows exactly its type, but this is unknown to the market. However, the market knows that 50% of the existing firms are of type H and 50% of type L. Both the firm and the market are risk neutral. Finally assume that the market can borrow at zero net interest rate.

   (a) Suppose the project is undertaken by all firms, irrespective of their type. Compute the market value of the shares belonging to the original shareholders and the market value of the shares held by the new shareholders under alternative realizations of uncertainty.

   (b) Suppose that the management of the firm maximizes original shareholders' wealth. What will be the equilibrium in this case? Which firms will undertake the project and which will not? Explain your answer.

   (c) Suppose now that the firm is run by managers maximizing the total value of the firm. Does the optimal choice change in this case?

   (d) How does the answer to question (b) change if the fraction of firms of type H and L is of $\frac{3}{4}$ and $\frac{1}{4}$, respectively?

2. **Mean variance analysis**

   (a) Consider two securities, say **a** and **b**, with expected returns $Er^a = 10\%$ and $Er^b = 20\%$, standard deviation $\sigma_a = 0.2$ and $\sigma_b = 0.3$ respectively. Suppose they constitute the market portfolio with proportions 0.4 and 0.6. Let their correlation coefficient equal to 0.3.
   Based on this information and given a riskfree rate of 5%, specify the equation for the Capital Market Line (CML).

   (b) In a standard mean-variance analysis world, is it possible for a security not to be part of the market portfolio? Explain.

3. **CAPM**

   (a) Show that under "standard" conditions, no investor would hold a portfolio, say p, where the trade off between expected return and standard deviation for any two assets, say **a** and **b**, is given by the following values: $\frac{\frac{\partial Er^p}{\partial X_a}}{\frac{\partial \sigma_p}{\partial X_a}} = \frac{5}{2}$, and $\frac{\frac{\partial Er^p}{\partial X_b}}{\frac{\partial \sigma_p}{\partial X_b}} = \frac{4}{2}$.

   (b) Distinguish between the CML and the SML.

   (c) Show that the beta of a portfolio is equal to the weighted average of the assets it contains.

   (d) Explain why the CAPM does not hold if (a) borrowing and lending rates are different across agents and (b) agents are asymmetrically informed on the returns of financial assets.